Chapter 2: Section 2.2 **The Limit of a Function**

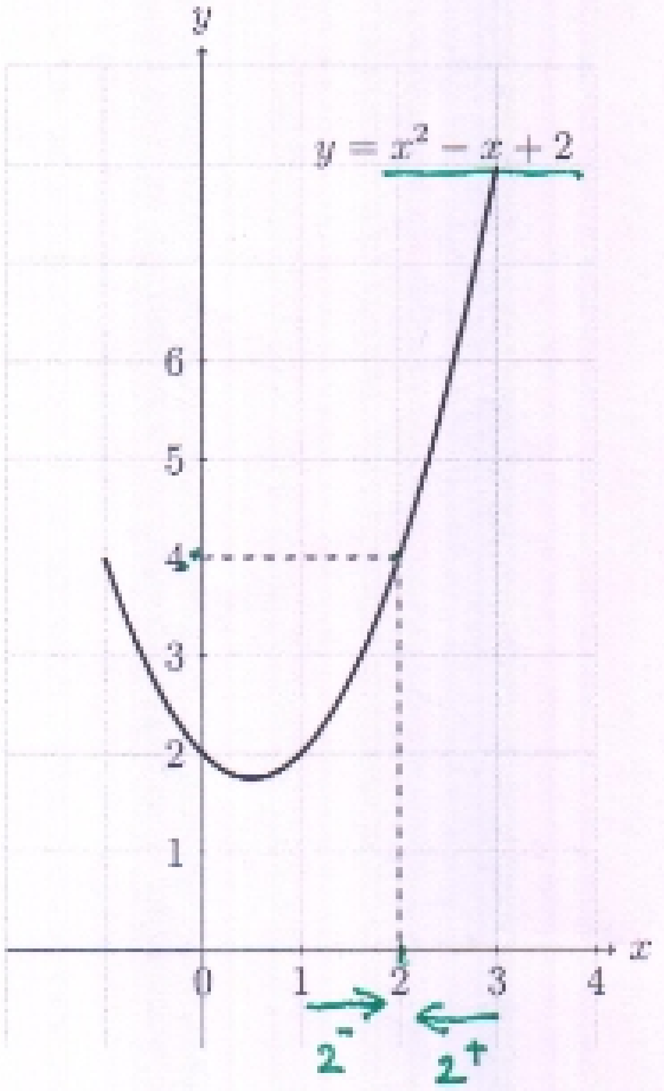
Definition: **The Limit of a Function,** We write

$$\lim_{x\to a} f(x) = L$$

and say **"the limit of $f(x)$, as x approaches a, equals L"** if we can make the values of $f(x)$ arbitrarily close to L(as close to L as we like) by taking x to be sufficiently close to a(on either side of a) but not equal to a.

Let us investigate the behavior of the function $f(x)$ defined by $f(x) = x^2 - x + 2$ for values x near to 2. The following table gives values of $f(x)$ for values of x close to 2 and finally $x = 2$.

$\lim_{x\to 2} f(x)$



x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001
2.0	$\boxed{4.000000}$	2.0	$\boxed{4.000000}$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 4$

Limit exist

From the table and graph of f (a parabola) shown in the above figure we see that when x is close to 2 (on either side of 2), $f(x)$ is close to 4. We express this by saying **"the limit of the function $\mathbf{f(x) = x^2 - x + 2}$ as x approaches 2 is equal to 4."**

Example 1: Explain what it means to say that

$$\lim_{x\to 1^-} f(x) = 3 \qquad \lim_{x\to 1^+} f(x) = 7$$

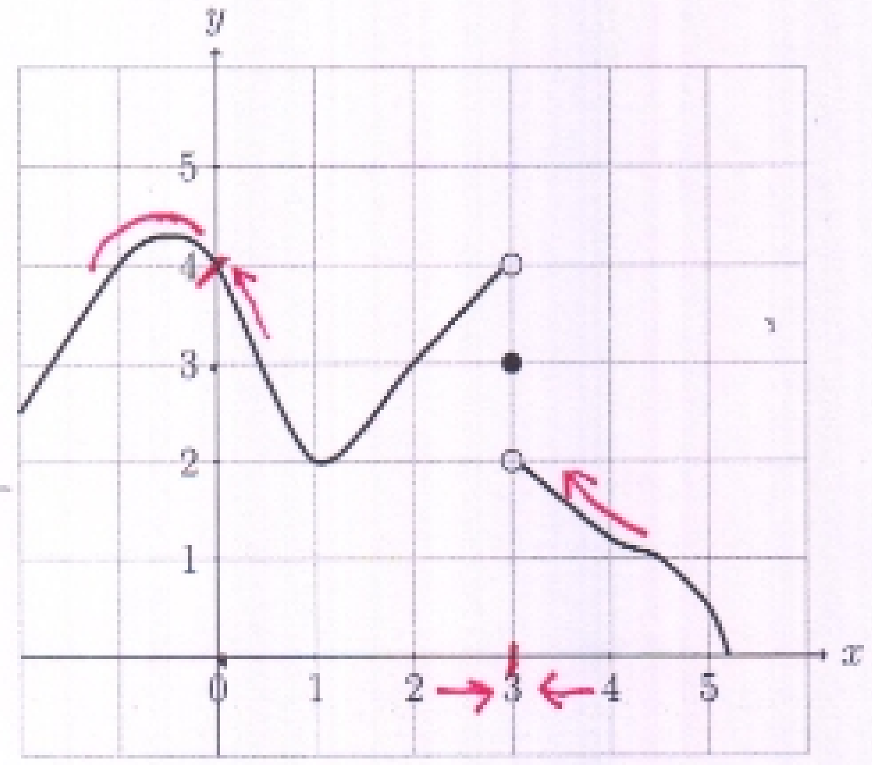
In this situation is it possible that $\lim_{x\to 1} f(x)$ exists? Explain.

$\text{LHL} = \lim_{x\to 1^-} f(x) = 3$

$\text{RHL} = \lim_{x\to 1^+} f(x) = 7$

$\Rightarrow \lim_{x\to 1} f(x) = \text{DNE}$

Example 2: For the function f whose graph is given, state the value of each quantity, if exists. If it does not exist, explain why?



(a) $\lim_{x \to 0} f(x) = 4$ LHL = 4
RHL = 4

(b) $\lim_{x \to 3^-} f(x) = 4$

(c) $\lim_{x \to 3^+} f(x) = 2$

(d) $\lim_{x \to 3} f(x) =$ DNE

(e) $f(3) = 3$

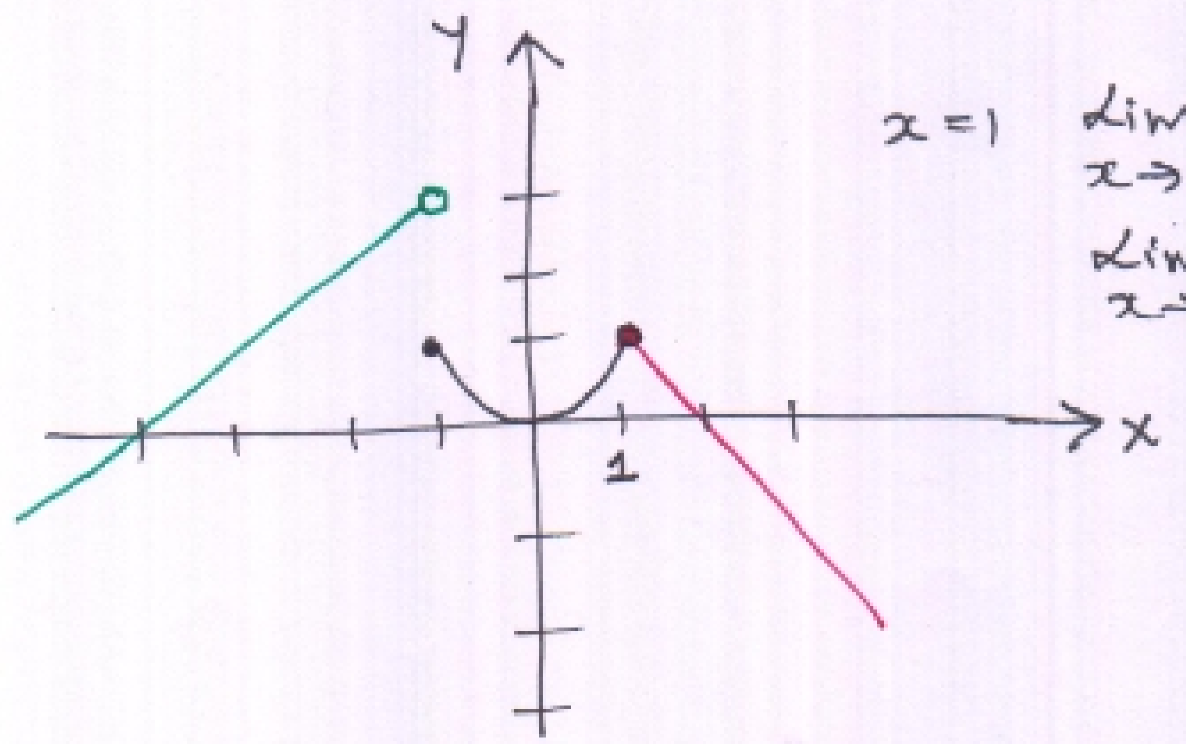
NOTE:

$$\boxed{\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x) = L}$$

Example 3: Sketch the graph of the function and use it to determine the values of a for which $\lim_{x \to a} f(x)$ exists.

$$f(x) = \begin{cases} 4 + x & \text{if} \quad x < -1 \\ x^2 & \text{if} \quad -1 \le x < 1 \\ 2 - x & \text{if} \quad x \ge 1 \end{cases}$$

→ Piece-wise function



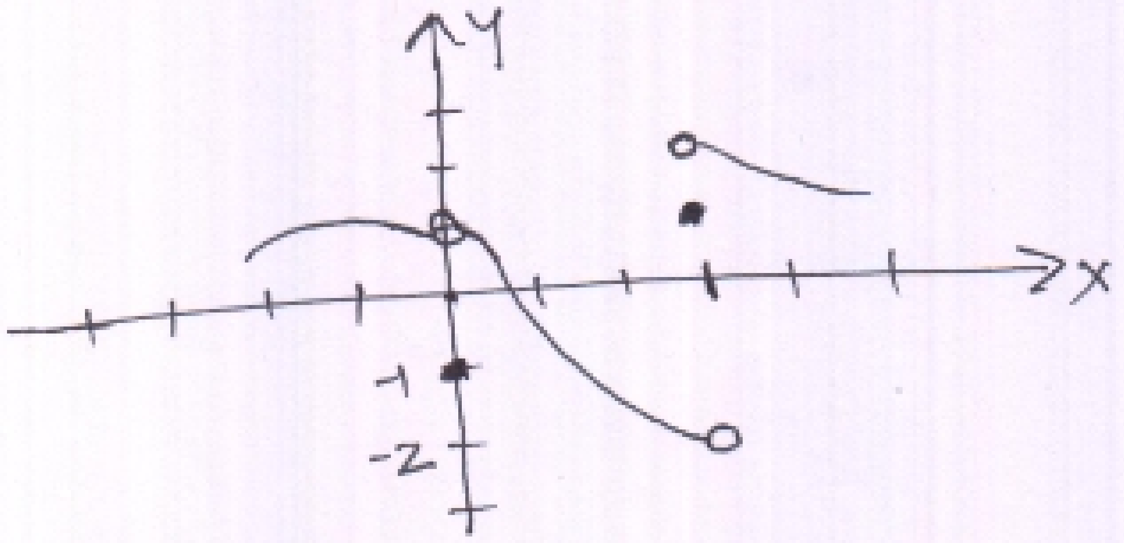
$x = 1$ $\lim_{x \to 1^-} f(x) = 1$

$\lim_{x \to 1^+} f(x) = 1$

$\therefore \lim_{x \to 1} f(x) = 1$

Example 4: Sketch the graph of an example of a function f that satisfies all of the given conditions:

$$\lim_{x\to 0} f(x) = 1, \quad \lim_{x\to 3^-} f(x) = -2, \quad \lim_{x\to 3^+} f(x) = 2, \quad f(0) = -1, \quad f(3) = 1.$$



Example 5: Use a table of values to estimate the value of the limit

$$\lim_{x\to 0} \frac{\sin(x)}{\sin(2x)}$$

0

$x = -0.001$ —— $\dfrac{\sin(x)}{\sin(2x)} \sim 0.5$

$x = 0.001$ —— $\dfrac{\sin(x)}{\sin(2x)} \sim 0.5$

$\therefore \lim_{x\to 0} \dfrac{\sin x}{\sin 2x} = 0.5$

Example 6: Use the graph of the function f to state value of each limit, if it exists.

$$f(x) = \frac{3}{1 + e^{1/x}}$$

(i) $\lim_{x\to 0^-} f(x) = 3$

(ii) $\lim_{x\to 0^+} f(x) = 0$

(iii) $\lim_{x\to 0} f(x) = \text{DNE}$