Chapter 2: Section 2.3 **Calculating Limits Using the Limit Laws**

Limit Laws: Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

$\lim_{x\to a^-} f(x) = L$

$\lim_{x\to a^+} f(x) = L$

exist. Then

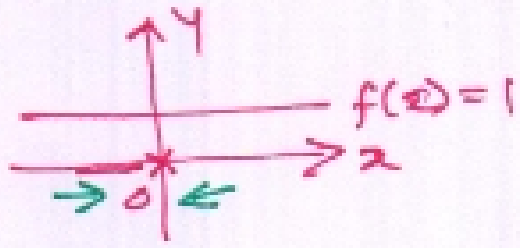
(1) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(2) $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(3) $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

(4) $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$

(5) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$.



(6) $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$ where n is a positive integer.

(7) $\lim_{x\to a} c = c$ $\quad \lim_{x\to 1} c = c$

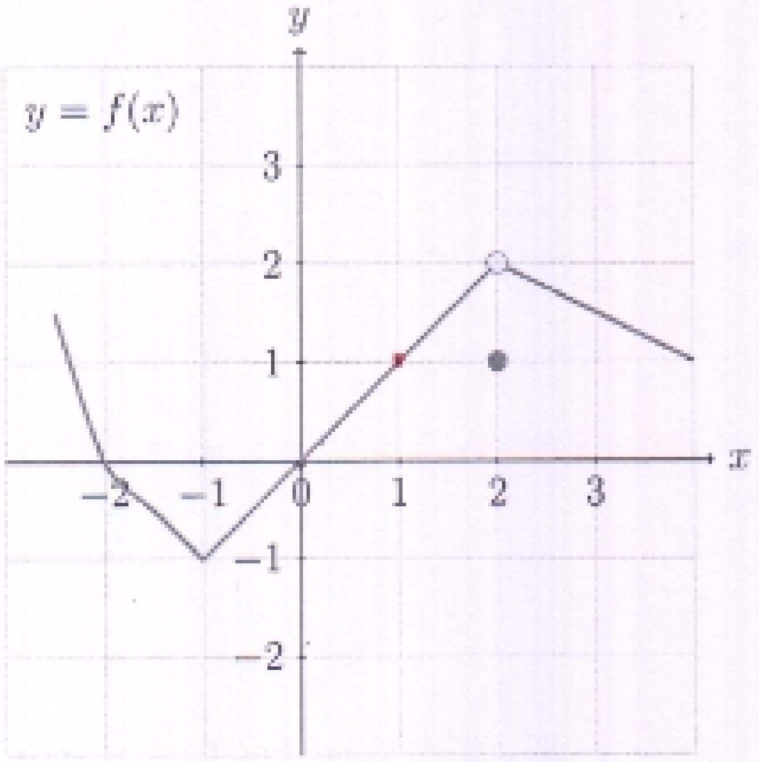
(8) $\lim_{x\to a} x = a$

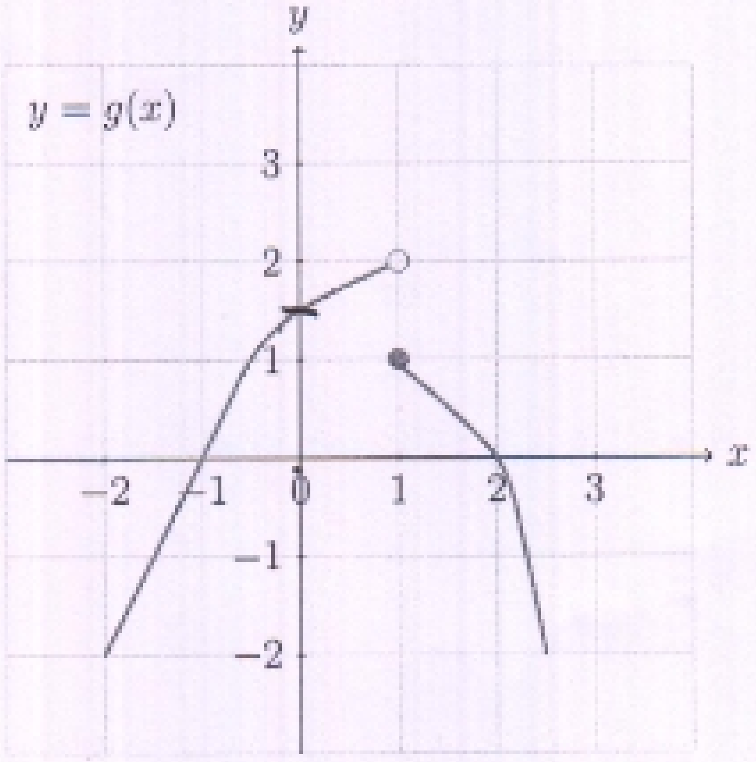
(9) $\lim_{x\to a} x^n = a^n$ where n is a positive integer.

(10) $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer. If n is even, we assume that $a > 0$.

(11) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer. If n is even, we assume that $\lim_{x\to a} > 0$.

Example 1: The graphs of f and g are given. Use them to evaluate each limit. If the limit does not exist, explain why.





(a) $\lim\limits_{x\to2}[f(x)+g(x)] = \lim\limits_{x\to2} f(x) + \lim\limits_{x\to2} g(x)$

$= 2 + 0 = 2$

(b) $\lim\limits_{x\to1}[f(x)+g(x)] = \lim\limits_{x\to1} f(x) + \underbrace{\lim\limits_{x\to1} g(x)}_{DNE} = DNE$

$= 1 + DNE$

(c) $\lim\limits_{x\to0}[f(x)\cdot g(x)]$

$= \lim\limits_{x\to0} f(x) \cdot \lim\limits_{x\to0} g(x) = 0 * \frac{3}{2} = 0$ $\boxed{\frac{1}{0} = \infty}$

(d) $\lim\limits_{x\to-1}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to-1} f(x)}{\lim\limits_{x\to-1} g(x)} = \frac{-1}{0} = DNE \sim -\infty$

(e) $\lim\limits_{x\to2}[x^3 f(x)] = \lim\limits_{x\to2} x^3 \cdot \lim\limits_{x\to2} f(x)$

$= 2^3 * 2 = 16$

(f) $\lim\limits_{x\to1}\sqrt{3+f(x)} = \sqrt{\lim\limits_{x\to1}(3+f(x))} = \sqrt{\lim\limits_{x\to1} 3 + \lim\limits_{x\to1} f(x)}$

$= \sqrt{3+1}$

$= \sqrt{4} = 2$

$\rightarrow \frac{f(x)}{g(x)}$

Direct Substitution Property: If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Example 2: Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

(a) $\lim_{x\to -2}(3x^4 + 2x^2 - x + 1) = \lim_{x\to -2} 3x^4 + \lim_{x\to -2} 2x^2 - \lim_{x\to -2} x + \lim_{x\to -2} 1$ (L1)

$= 3\lim_{x\to -2} x^4 + 2\lim_{x\to -2} x^2 - \lim_{x\to -2} x + \lim_{x\to -2} 1$ (L3)

$= 3(-2)^4 + 2(-2)^2 - (-2) + 1$ (L7 & L9)

$= 59$

(b) $\lim_{x\to 2}\sqrt{\frac{2x^2+1}{3x-2}} = \sqrt{\lim_{x\to 2}\frac{2x^2+1}{3x-2}} = \sqrt{\frac{\lim_{x\to 2} 2x^2+1}{\lim_{x\to 2} 3x-2}} = \sqrt{\frac{2(2)^2+1}{3(2)-2}}$

$= \sqrt{\frac{9}{4}} = \frac{3}{2}$

Example 3: Evaluate the limit, if it exists.

(a) $\lim_{x\to 4}\frac{x^2-4x}{x^2-3x-4} = \frac{4^2-4(4)}{4^2-3(4)-4} = \frac{16-16}{16-12-4} = \frac{0}{0}$ = INDETERMINANT FORM

$= \lim_{x\to 4}\frac{x\cancel{(x-4)}}{\cancel{(x-4)}(x+1)} = \lim_{x\to 4}\frac{x}{x+1} = \frac{4}{4+1} = \frac{4}{5}$

(b) $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{0+1}-1}{0} = \frac{\sqrt{1}-1}{0} = \frac{1-1}{0} = \frac{0}{0}$

$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x} \times \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \lim_{x\to 0}\frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)}$

$= \lim_{x\to 0}\frac{x}{x(\sqrt{x+1}+1)} = \lim_{x\to 0}\frac{1}{\sqrt{x+1}+1} = \frac{1}{\sqrt{0+1}+1} = \frac{1}{2} = \frac{1}{2}$

(c) $\lim_{x\to -4}\frac{\frac{1}{4}+\frac{1}{x}}{4+x}$

$= \frac{\frac{1}{4}+\frac{1}{-4}}{4+(-4)} = \frac{\frac{1}{4}-\frac{1}{4}}{4-4} = \frac{0}{0}$

$\lim_{x\to -4}\frac{\frac{x}{4x}+\frac{4}{4x}}{4+x} = \lim_{x\to -4}\frac{\frac{x+4}{4x}}{4+x} = \lim_{x\to -4}\frac{x+4}{4x} \div \frac{4+x}{1}$

$= \lim_{x\to -4}\frac{\cancel{x+4}}{4x} * \frac{1}{\cancel{4+x}} = \lim_{x\to -4}\frac{1}{4x} = \frac{1}{4(-4)} = -\frac{1}{16}$