Inference for the Mean of a Population

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Inference for the Mean of a Population

We have already seen that when we take a SRS, $x_1, x_2, \ldots, x_n$, from a population with unknown μ and known σ, if either

- the population is Normally distributed, or
- n is large enough,

then

$$\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

Equivalently, the standardized sample mean, or the one-sample z statistic is

$$z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

What if the population SD σ is unknown? (A far more likely occurrence.)

Student's t Distribution

If the true population SD, σ, is unknown, we estimate σ using the sample SD S.

When the standard deviation of a statistic is estimated from the data, the result is called the standard error of the statistic. The standard error of the sample mean $\bar{x}$ is $SE_{\bar{X}} = \frac{S}{\sqrt{n}}$.

Now, instead of dealing with

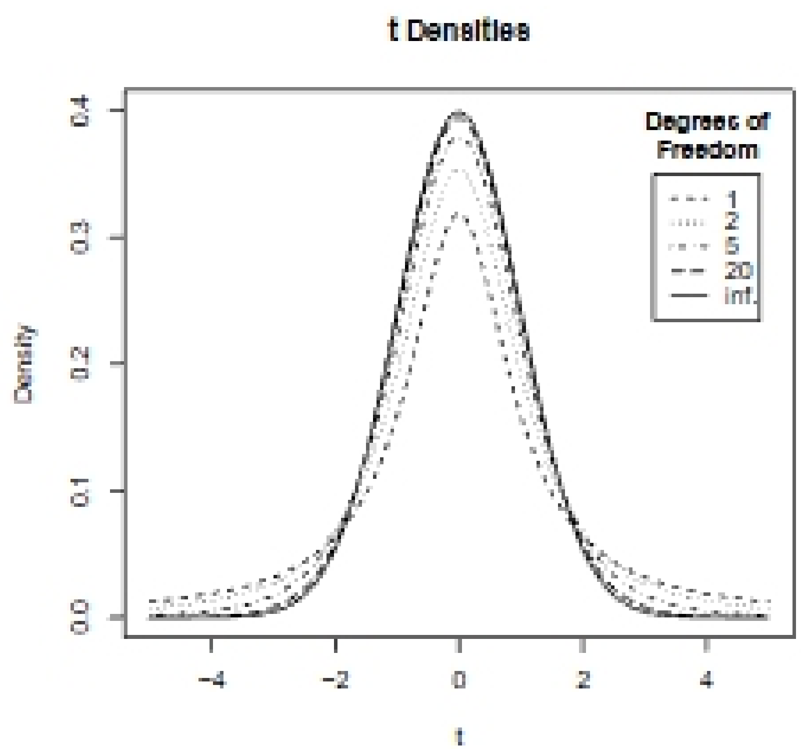
$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

we are interested in the quantity

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{(n-1)}$$

Here, $t_{(n-1)}$ is **Student's t distribution**, with $n-1$ *degrees of freedom*. The table for the t distribution is given in the back of the book. (Unlike the Normal or Binomial distributions, each of which has two parameters, the t distribution has only one parameter, called the **degrees of freedom**.)

The Density of Student's t



Properties of the t Distribution

- Symmetric about zero
- Bell-shaped – similar to normal distribution
- More spread out than normal – heavier tails
- Exact shape depends on the degrees of freedom
- As the number of degrees of freedom increases, the t distribution converges to the Normal distribution.

Confidence Intervals with Unknown σ

Recall that, for a population with unknown μ and *known* σ, $100(1-\alpha)\%$ CI for μ, based on an SRS $x_1, x_2, \ldots, x_n$ is given by

$$\left(\bar{x} - z^* \frac{\sigma}{\sqrt{n}},\ \bar{x} + z^* \frac{\sigma}{\sqrt{n}}\right)$$

If σ is unknown,

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{(n-1)}$$

so we substitute a t-critical value, t^* for z^*:

$$\left(\bar{x} - t^* \frac{s}{\sqrt{n}},\ \bar{x} + t^* \frac{s}{\sqrt{n}}\right)$$

The critical value, t^*, is chosen such that $100(1-\alpha)\%$ of the area under the $t_{(n-1)}$ density lies between $-t^*$ and t^*.

The area to the right of t^* should be $\frac{\alpha}{2}$ and the area to the left of $-t^*$ should be $\frac{\alpha}{2}$.

What would t^* be for a 95% confidence interval for a t distribution with 30 degrees of freedom, i.e. $n = 31$?

For this question, $\alpha = .05$. For a t_{30} density, the area to the right of 2.042 is $\frac{.05}{2} = .025$ and the area to the left of -2.042 is .025, so $t^* = 2.042$

For a 95% confidence interval for a t distribution with 100 degrees of freedom , $t^* = 1.984$.

For a 95% confidence interval for a t distribution with 1000 degrees of freedom, $t^* = 1.962$.

Note that a $z^* = 1.960$ for a 95% confidence interval for a standard normal distribution.

If the underlying population is Normally distributed, the confidence interval

$$\left(\bar{x} - t^* \frac{s}{\sqrt{n}},\ \bar{x} + t^* \frac{s}{\sqrt{n}}\right)$$

is exact.

Otherwise, if n is not too small ($n \geq 15$), the data are not strongly skewed, and there are no outliers, the interval is approximately correct.

With n sufficiently large (say $n \geq 40$), the approximation is correct even if the data are clearly skewed.

A One-sample t-test

Suppose a SRS of size n is drawn from a $N(\mu, \sigma)$ population with both μ and σ unknown. The t-statistic,

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has the t distribution with $n - 1$ d.f.

To test $H_0 : \mu = \mu_0$, compute the one-sample t statistic,

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

The p-values are

$$\begin{array}{ll} H_a : \mu > \mu_0 & P(t_{n-1} \geq t) \\ H_a : \mu < \mu_0 & P(t_{n-1} \leq t) \\ H_a : \mu \neq \mu_0 & 2P(t_{n-1} \geq |t|) \end{array}$$

These are exact if the population is normal, and otherwise approximately correct for large n.

Example

Let X (in mm) denote the growth in 15 days of a tumor induced in a mouse. It is known from a previous experiment that the average tumor growth is $4mm$. A sample of 20 mice that have a genetic variant hypothesized to be involved in tumor growth yielded $\bar{x} = 3.8mm$, $s = 0.3mm$. Test whether $\mu = 4$ or not, assuming growths are normally distributed.

1. State the hypotheses

$$H_0 : \mu = 4 \qquad H_a : \mu \neq 4$$

2. Calculate the t-statistic

$$t = \frac{3.8 - 4.0}{0.3/\sqrt{20}} = -2.98$$

3. Determine the p-value using the t-distribution table in the book

$$p = 2\text{P}(t_{19} \geq 2.98) < 2\text{P}(t_{19} \geq 2.861) = 2(.005) = .01$$

Since p is less than 0.01, we reject H_0 at significance level $\alpha = 0.01$. (p-value=0.008)

There is evidence that the population mean tumor growth is not $4mm$.

What if we wanted a 99% CI for μ instead? For this case $\alpha = .01$ We need to first need to find t^* for a t distribution with 19 degrees of freedom. What is t^*?

The CI is given by

$$\left(\bar{x} - t^* \frac{s}{\sqrt{n}},\ \bar{x} + t^* \frac{s}{\sqrt{n}}\right) =$$

$$= \left(3.8 - 2.861 \times \frac{0.3}{\sqrt{20}},\ 3.8 + 2.861 \times \frac{0.3}{\sqrt{20}}\right) =$$

$$= (3.61,\ 3.99)$$

where $t^* = 2.861$ is the upper 0.005 critical value of t_{19}.

Note that 4 is outside this CI. From this, we can draw the same conclusion as from the test. Namely, at significance level $\alpha = 0.01$, the mean growth not equal to $4mm$.

Confidence Intervals and Two-Sided Tests

A two-sided hypothesis test with significance level α rejects the null hypothesis $H_0 : \mu = \mu_0$ if and only if the value μ_0 falls outside the $100(1 - \alpha)$% CI for μ.

Reporting a CI is generally more informative than just reporting a p-value or the decision made on the basis of a hypothesis test.

One-Sided Alternatives

In the previous example, suppose we wished to test whether $\mu < 4$.

1. State the hypotheses

$$H_0 : \mu = 4 \qquad H_a : \mu < 4$$

2. Calculate the t-statistic

$$t = \frac{3.8 - 4}{0.3/\sqrt{20}} = -2.98$$