Equations solvable by direct integration

Last time: Newton's Law of cooling

$$\frac{dT}{dt} = -k(T-A)$$

How to solve? One asks, why not just integrate with respect to t?

$$T = \int \frac{dT}{dt}\,dt = \underbrace{\int -k(T-A)\,dt}$$

This is true ... but here we are integrating the unknown function $T(t)$, so we can't do this integral. **FAIL**

We failed because the Right hand side of the D.E. depends on the unknown function T

$$\frac{dT}{dt} = -k(T-A)$$

But, this strategy will work if the right hand side doesn't depend on the unknown function.

Let's formulate this case precisely

Notation: x — independent variable
y — dependent variable
$y(x)$ - <u>unknown function</u> we wish to find

Consider D.E. of the form

$$\boxed{\frac{dy}{dx} = f(x)}$$

Where $f(x)$ is some given function

Eg $\frac{dy}{dx} = 7x^2 + 4$, $\frac{dy}{dx} = e^{2x}$, $\frac{dy}{dx} = \arctan(x)$

We can (at least theoretically) integrate both sides dx to get solution.

<u>Eg</u> $$\frac{dy}{dx} = 7x^2 + 4$$

$$\int \frac{dy}{dx}\,dx = \int (7x^2+4)\,dx$$

Fund. Thm. Calc.

$$y = \frac{7}{3}x^3 + 4x + C$$

Because we are doing indefinite integrals, we need to put a constant of integration in this equation

In abstract terms: For the equation $\dfrac{dy}{dx} = f(x)$

General solution is $\quad y(x) = \int f(x)\,dx + C$

The constant of integration is not just some pedantic thing any more: We need it in order to satisfy an initial condition

We need the initial condition because the differential equation tells us how y changes, but it doesn't tell us where y starts

Consider problem $\begin{cases} \dfrac{dy}{dx} = 7x^2 + 4 \\ y(0) = 9 \end{cases}$

We showed $\quad y(x) = \frac{7}{3}x^3 + 4x + C \quad$ for some constant C

Plug in $x=0 \quad y(0) = \frac{7}{3}0^3 + 4 \cdot 0 + C = C$

So set $\boxed{C = 9}$

$y(x) = \frac{7}{3}x^3 + 4x + 9$ solves $\begin{cases} \dfrac{dy}{dx} = 7x^2 + 4 \\ y(0) = 9 \end{cases}$

This is called a particular solution