Separable equations

Today we will finally learn how to solve the equation

$$\frac{dy}{dx} = xy$$

Can't integrate directly, but what if we rearrange first?

Multiply by $\frac{1}{y}$: $\frac{1}{y}\frac{dy}{dx} = x$

Now integrate both sides $\int dx$:

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int x\,dx = \frac{1}{2}x^2 + C$$

As for $\int \frac{1}{y}\frac{dy}{dx}\,dx$, it equals $\int \frac{1}{y}\,dy$

Why? (1) This is a type of u-substitution, with $u = y$

Consider $u = y$, $du = \frac{dy}{dx}dx$ so $\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{u}\,du = \int \frac{1}{y}\,dy$

Equivalently, we can use the chain rule and the fundamental theorem of calculus:

$$\frac{d}{dx}\left(\int \frac{1}{y}\,dy\right) \overset{\text{Chain rule}}{=} \frac{d}{dy}\left(\int \frac{1}{y}\,dy\right)\frac{dy}{dx} \overset{\text{FTC}}{=} \frac{1}{y}\frac{dy}{dx}$$

So $\int \frac{1}{y}\,dy = \int \frac{1}{y}\frac{dy}{dx}\,dx$

Either way, we know $\int \frac{1}{y}\, dy = \ln|y|$.

So we get
$$\ln|y| = \frac{1}{2}x^2 + C$$

The last step is to solve for y as a function of x.

Exponentiate $\quad |y| = e^{\left(\frac{1}{2}x^2 + C\right)} = e^{\frac{1}{2}x^2} e^{C} = e^{C} e^{\frac{1}{2}x^2}$

Remove absolute value bars! $\quad y = \pm e^{C} e^{\frac{1}{2}x^2}$

Note that $\pm e^{C}$ is just a constant, call it D:
$$y = De^{\frac{1}{2}x^2}$$

Check your solution:
$$\frac{dy}{dx} = D \frac{d}{dx}\left(e^{\frac{1}{2}x^2}\right) = De^{\frac{1}{2}x^2} \frac{d}{dx}\left(\frac{1}{2}x^2\right) = De^{\frac{1}{2}x^2} x = yx$$

The solution is good.

To summarize: $y(x) = De^{\frac{1}{2}x^2}$ is the general solution of $\frac{dy}{dx} = xy$.

A quicker notation uses differentials

$$\frac{dy}{dx} = xy$$

$$\frac{1}{y}\,dy = x\,dx$$

$$\int \frac{1}{y}\,dy = \int x\,dx \quad \longrightarrow \quad \ln|y| = \frac{1}{2}x^2 + C$$

and so on as before.

Another example: Newton's law of cooling

$$\frac{dT}{dt} = -k(T-A)$$

Separate: $$\frac{1}{T-A}\,dT = -k\,dt$$

Integrate $$\int \frac{1}{T-A}\,dT = \int -k\,dt = -kt + C$$

To do $\int \frac{1}{T-A}\,dT$, substitute $u = T-A$, $du = dT$

$$\int \frac{1}{u}\,du = \ln|u| = \ln|T-A|$$

We get $\ln|T-A| = -kt + C$

$$|T-A| = e^{-kt}e^{C}$$

$$T-A = \pm e^{C}e^{-kt} \quad \longrightarrow \quad T-A = De^{-kt}$$

$$T = A + De^{-kt}$$

D = arbitrary constant.