First Order Linear equations (integrating factors)

This is an important and rather general class of equations that can be solved in terms of integrals

<u>Definition</u> a <u>first-order linear equation</u> is one that can be written as

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$P(x)$ and $Q(x)$ are the <u>coefficient functions</u>.

Call <u>Linear</u> because $\frac{dy}{dx}$ and y appear to first power ONLY.

<u>Eg</u> $\frac{dy}{dx} = xy \iff \frac{dy}{dx} - x\,y = 0$ $\quad P(x) = -x$, $Q(x) = 0$

$\frac{dT}{dt} = -k(T-A) \iff \frac{dT}{dt} + kT = kA$ $\quad P(t) = k$, $Q(t) = kA$

$x\frac{dy}{dx} - y = x^3 \iff \frac{dy}{dx} - \frac{1}{x}\,y = x^2$ $\quad P(x) = -\frac{1}{x}$, $Q(x) = x^2$

* We always want the equation in STANDARD FORM

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

To solve, we want to be able to integrate LHS, meaning we need to <u>recognize it as a derivative.</u>

The trick is to multiply the <u>whole equation</u> by a function $\rho(x)$ first:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\underbrace{\rho(x)\frac{dy}{dx} + \rho(x)P(x)y}_{} = \rho(x)\,Q(x)$$

This is reminiscent of the product rule

$$\frac{d}{dx}\left(\rho(x)y\right) = \rho(x)\frac{dy}{dx} + \frac{d\rho}{dx}y$$

What if we pick $\rho(x)$ so that $\frac{d\rho}{dx} = \rho(x)\,P(x)$?

This equation is separable:

$$\frac{1}{\rho}\frac{d\rho}{dx} = P(x) \longrightarrow \int \frac{1}{\rho}\,d\rho = \int P(x)\,dx$$

$$\text{so } \ln|\rho| = \int P(x)\,dx$$

$$\rho = Ce^{\int P(x)\,dx}$$

We only need one such ρ, so we just take

$$\rho(x) = e^{\int P(x)\,dx}$$

[This happens to be a situation where the constant of integration doesn't matter.]

Going back to the equation, with $\rho(x) = e^{\int P(x)dx}$

$$e^{\int P(x)dx}\frac{dy}{dx} + e^{\int P(x)dx}P(x)\,y = e^{\int P(x)dx}Q(x)$$

$$\frac{d}{dx}\left(e^{\int P(x)dx}y\right) = e^{\int P(x)dx}Q(x)$$

Can finally integrate!

$$e^{\int P(x)dx}y = \int\left(e^{\int P(x)dx}Q(x)\right)dx + C$$

Now the constant does matter.

Solve for y:

$$y = e^{-\int P(x)dx}\int\left(e^{\int P(x)dx}Q(x)\right)dx + Ce^{-\int P(x)dx}$$

Let's see it in action

$$\frac{dy}{dx} + 3y = 2xe^{-3x}$$

$P(x) = 3$
$Q(x) = 2xe^{-3x}$

Step 1 find integrating factor $e^{\int P(x)dx}$ $\quad \int 3dx = 3x$

So we can use $\boxed{e^{3x}}$