

Substitution and other methods.

Just as  $u$ -substitution sometimes makes integrals easier, we can use it to solve differential equations

$$\frac{dy}{dx} = (4x + y)^2$$

Try  $u = 4x + y$        $\frac{du}{dx} = 4 + \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{du}{dx} - 4$$

$$\frac{du}{dx} - 4 = u^2 \quad \frac{du}{dx} = u^2 + 4$$

$$\int \frac{du}{u^2 + 4} = \int 1 dx = x + C$$

$$\int \frac{du}{u^2 + 4} = \frac{1}{2} \arctan\left(\frac{u}{2}\right)$$

$$\Rightarrow \frac{1}{2} \arctan\left(\frac{u}{2}\right) = x + C$$

$$u = 2 \tan(2x + 2C)$$

$$4x + y = 2 \tan(2x + 2C)$$

$$y = 2 \tan(2x + 2C) - 4x.$$

Sometimes one must manipulate the equation to find a good substitution

$$x^2 \frac{dy}{dx} = xy + x^2 e^{y/x}$$

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

Try  $u = \frac{y}{x}$        $\frac{du}{dx} = \frac{-y}{x^2} + \frac{1}{x} \frac{dy}{dx}$

$$\frac{dy}{dx} = x \frac{du}{dx} + \frac{y}{x} = x \frac{du}{dx} + u$$

$$x \frac{du}{dx} + u = u + e^u$$

$$x \frac{du}{dx} = e^u \rightarrow \frac{du}{e^u} = \frac{dx}{x}$$

$$-e^{-u} = \ln|x| + C$$

$$-u = \ln(-\ln|x| - C)$$

$$u = -\ln(-\ln|x| - C)$$

$$\frac{y}{x} = -\ln(-\ln|x| - C)$$

$$y = -x \ln(-\ln|x| - C)$$

Bernoulli equation is another standard form with a standard substitution.

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

if  $n=0$  or  $n=1$   
this is a linear equation  
use integrating factor.

Use the substitution  $u = y^{1-n}$  for a Bernoulli equation.

Eg  $x \frac{dy}{dx} + y = 3xy^{4/3}$

$n = \frac{4}{3}$ ,  $1-n = -\frac{1}{3}$  so use  $u = y^{-1/3}$   $y = u^{-3}$

$$\frac{dy}{dx} = -3u^{-4} \frac{du}{dx}$$

$$-3xu^{-4} \frac{du}{dx} + u^{-3} = 3xu^{-4}$$

Divide by  $-3xu^{-4}$   $\frac{du}{dx} - \frac{1}{3x}u = -1$

This is a linear equation  $\int P(x)dx = \int \frac{-1}{3x} dx = -\frac{1}{3} \ln x$

Integrating factor:  $f(x) = e^{-\frac{1}{3} \ln x} = x^{-\frac{1}{3}}$

$$x^{-\frac{1}{3}} \frac{du}{dx} - \frac{1}{3} x^{-\frac{4}{3}} u = -x^{-\frac{1}{3}}$$

$$\frac{d}{dx} (x^{-\frac{1}{3}} u) = -x^{-\frac{1}{3}}$$