Exact equations cont'd; Population models.

Recall: For any unknown function $y(x)$, the conditions

① $F(x,y) = C$

and

② $\frac{\partial F}{\partial x}(x,y) + \frac{\partial F}{\partial y}(x,y)\frac{dy}{dx} = 0$

are equivalent.

An <u>exact differential equation</u> is one that looks like ②

<u>Definition</u> An equation of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is <u>exact</u> if there is a function $F(x,y)$ such that

$$M(x,y) = \frac{\partial F}{\partial x}(x,y) \quad \text{and} \quad N(x,y) = \frac{\partial F}{\partial y}(x,y).$$

Eg.

$$(4x-y) + (6y-x)\frac{dy}{dx} = 0$$

$$M(x,y) = 4x-y, \quad N(x,y) = 6y-x$$

$\frac{\partial F}{\partial x} = M$ implies $F = \int M\,dx = \int (4x-y)\,dx = 2x^2 - xy + C(y)$

This "constant" of integration may depend on y!

Need $\frac{\partial F}{\partial y} = N$ or $-x + \frac{dC}{dy}(y) = 6y - x$

So $\frac{dC}{dy} = 6y \Rightarrow C = \int 6y\,dy = 3y^2 + D$

This is a true constant.

So $F(x,y) = 2x^2 - xy + 3y^2 + D$

Check again: $\frac{\partial F}{\partial x} = 4x - y = M$

$\frac{\partial F}{\partial y} = -x + 6y = N$

Solutions of DE satisfy $2x^2 + xy + 3y^2 = 0$ (use quadratic formula to solve for y)

This process won't <u>always</u> work:

<u>Observe</u>: if $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$

Then $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\frac{\partial F}{\partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial F}{\partial y} = \frac{\partial N}{\partial x}$

So $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is a <u>necessary condition</u> for the equation to be exact.

In the previous example $M = 4x - y$

$N = 6y - x$

$\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = -1$, so it is consistent.

<u>Non-example:</u> $\sin(x)y^2 + 8\frac{dy}{dx} = 0$

$M = \sin(x)\,y^2$ $\qquad \frac{\partial M}{\partial y} = 2\sin(x)\,y$

$N = 8$

$\frac{\partial N}{\partial x} = 0$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, there is <u>no function</u> $F(x,y)$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$.

<u>Test for exactness:</u> $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

<u>Eg</u> $\left(x^3 + \frac{y}{x}\right) + \left(y^2 + \ln x\right)\frac{dy}{dx} = 0$

$M = x^3 + \frac{y}{x}$ $\qquad N = y^2 + \ln x$

$\frac{\partial M}{\partial y} = \frac{1}{x}$ $\qquad \frac{\partial N}{\partial x} = \frac{1}{x}$ $\longrightarrow$ so it is exact!

To find F: $F(x,y) = \int \left(x^3 + \frac{y}{x}\right) dx = \frac{1}{4}x^4 + y\ln x + C(y)$

$y^2 + \ln x = \frac{\partial F}{\partial y} = \ln x + \frac{dC}{dy}(y)$

So $y^2 = \frac{dC}{dy}$ and $C(y) = \frac{1}{3}y^3 + D$