Population models (examples of Autonomous DEs)

Let t denote time, and let $P(t)$ denote the population of some type of organism. (bacteria, humans, deer, ...)

We introduce

β = birth rate = births per unit time per unit of population.

δ = death rate

Over an interval of time Δt, there will be

$\beta \cdot P \cdot \Delta t$ births

and

$\delta \cdot P \cdot \Delta t$ deaths

so the change in population will be

$$\Delta P \approx \beta \cdot P \cdot \Delta t - \delta \cdot P \cdot \Delta t = (\beta - \delta) P \Delta t$$

$$\frac{\Delta P}{\Delta t} \approx (\beta - \delta) P$$

In limit as $\Delta t \to 0$, we obtain $\frac{dP}{dt} = (\beta - \delta) P$

This is the basic population model. We can vary the parameters β and δ, even allowing them to depend on t or P.
The point is to understand how the behavior of the population over time depends on the choice of parameters β, δ in the construction of the model.

Simplest case: β and δ are constant.

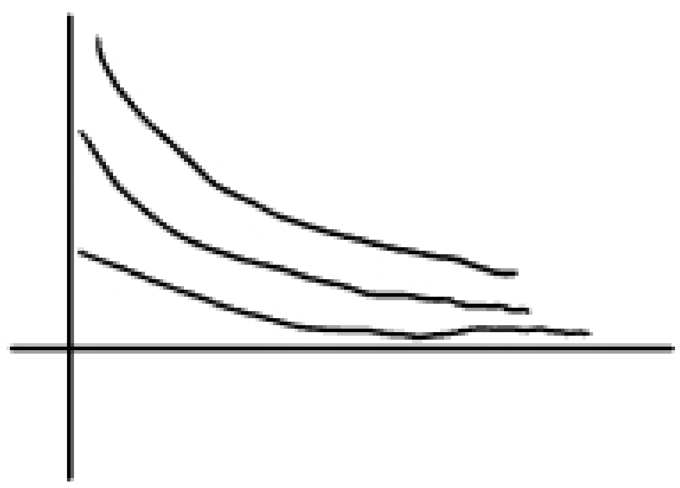
$$\frac{dP}{dt} = kP, \quad \text{where} \quad k = \beta - \delta$$

Solution: $P(t) = P_0 e^{kt}$ where $P_0 = P(0)$ is initial pop.

If $\beta > \delta$ then $k > 0$, and we have exponential growth.

EXPLOSION!

If $\beta < \delta$ then $k < 0$, and we have exponential decay

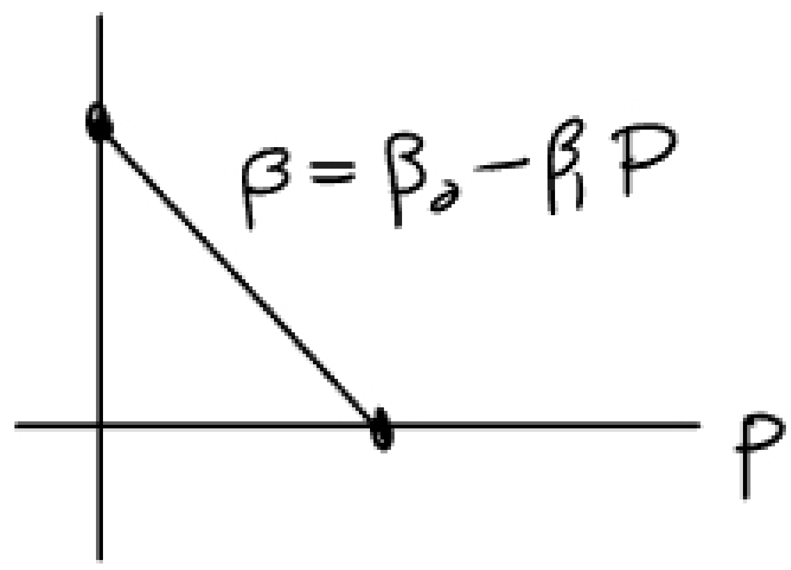


EXTINCTION!

What if β and δ depend on P?

A reasonable model is that the birth rate decreases with increased population. This will be true if there are environmental factors, such as a limited food supply, which limit reproduction at high population levels.

Then $\beta = \beta(P)$ will look like



Negative slope $-\beta_1$
Hits 0 at
$P = \dfrac{\beta_0}{\beta_1}$

Assume Death rate is constant $\delta = \delta_0$

So our population model becomes.

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta_0) P = (\beta_0 - \delta_0) P - \beta_1 P^2$$
$$= \beta_1 P \left(\frac{\beta_0 - \delta_0}{\beta_1} - P \right)$$

Or $\underbrace{\dfrac{dP}{dt} = k P (M - P)}$ $\qquad k = \beta_1$, $\quad M = \dfrac{\beta_0 - \delta_0}{\beta_1}$

This is called the <u>logistic equation</u>.

Many different situations are modeled by this equation