Second order linear equations continued.

Without further delay, let's see how to solve a 2nd order linear homogeneous equation.

We can do this when the equation has constant coefficients

Consider $ay'' + by' + cy = 0$

What are some solutions?

Recall 1st order case $ay' + by = 0$

$$\frac{y'}{y} = -b/a \longrightarrow \int \frac{dy}{y} = \int \left(-\frac{b}{a}\right) dx$$

$$\ln|y| = \left(-\frac{b}{a}\right) x + C \longrightarrow y = D e^{\left(-\frac{b}{a}\right) x}$$

So, let's try $y(x) = e^{rx}$, where $r$ is a constant to be determined.

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}$$

So $ay'' + by' + cy = 0$ becomes

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

Since $e^{rx}$ is never zero, we can divide by it:

$$ar^2 + br + c = 0.$$

This is the characteristic equation. It is the condition that $r$ must satisfy in order for $y = e^{rx}$ to solve the original DE.

Example: $$y'' - 4y' + 3y = 0$$

Try $e^{rx}$ $$r^2 e^{rx} - 4re^{rx} + 3e^{rx} = 0$$
$$r^2 - 4r + 3 = 0$$
$$(r-3)(r-1) = 0$$
so $r = 3$ or $1$.

Thus $y_1 = e^{3x}$ and $y_2 = e^{1x} = e^x$ are solutions!

Now by the principle of superposition for linear homogeneous equations, we know that
$$y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{3x} + c_2 e^{x}$$
is also a solution, for any constants $c_1$ and $c_2$.

In general, if $r_1$ and $r_2$ are solutions of the characteristic equation $ar^2 + br + c = 0$, then
$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$
is a solution of
$$ay'' + by' + cy = 0$$

And the solutions are real and distinct, The question is, how do we know when we've found "enough" solutions?

We start with a theoretical fact

<u>Theorem</u> (Existence and uniqueness for 2nd order linear equations)
Suppose $p(x)$, $q(x)$, and $f(x)$ are continuous on an interval $I$ and let $a$ be a point of $I$.

THEN the initial value problem

$$\begin{Bmatrix} y'' + p(x)y' + q(x)y = f(x) \\ y(a) = b_0 \\ y'(a) = b_1 \end{Bmatrix}$$

has a unique solution defined on $I$.

Rephrasing, we should get a unique solution for any <u>pair</u> of initial value and initial derivative.

We have found "enough" solutions when we have enough constants to solve any such initial value problem.

Suppose we find two solutions $y_1(x)$ and $y_2(x)$ to the homogeneous equation $y'' + p(x)y' + q(x)y = 0$.

Need to be able to solve for
$$y(a) = b_0$$
$$y'(a) = b_1$$

or
$$\begin{Bmatrix} c_1 y_1(a) + c_2 y_2(a) = b_0 \\ c_1 y_1'(a) + c_2 y_2'(a) = b_1 \end{Bmatrix}$$