

Nonhomogeneous linear equations, constant coefficient equations

[First we finish showing example of linear independence]

Nonhomogeneous linear diff. eqns.

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_{n-1}(x)y' + p_n(x)y = f(x)$$

Eg  $y'' - 4y = \sin(x)$

We cannot solve using characteristic equation.

Try:  $y = e^{rx}$        $r^2 e^{rx} - 4e^{rx} = \sin(x)$  ? can't work

Later, in section 3.5, we will learn a technique to solve this equation. It turns out that the solution is

$$y_p(x) = -\frac{1}{5} \sin(x)$$

Check:  $-\frac{1}{5}(-\sin(x)) - 4\left(-\frac{1}{5}\right)\sin(x) = \frac{\sin(x) + 4\sin(x)}{5} = \sin(x)$

Recall: The solutions of  $y'' - 4y = 0$

are  $y_1 = e^{2x}$ ,  $y_2 = e^{-2x}$ , with general solution

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

So the solution of the nonhomogeneous equation looks completely different from that of the homogeneous one.

The principle of superposition does not hold for nonhomogeneous equations.  
so for example  $2 \cdot \left(-\frac{1}{5} \sin(x)\right)$  is not a solution

But a related principle does apply

If we add a solution of the homogeneous equation to a solution of the nonhomogeneous equation, we get a solution of the nonhomogeneous equation

Let  $y_p$  be a solution of the nonhomogeneous equation  
 $y'' + p(x)y' + q(x)y = f(x)$  "particular solution"

So let  $y_c$  be a solution of the homogeneous equation  
 $y'' + p(x)y' + q(x)y = 0$

Then  $y_c + y_p$  satisfies  $y'' + p(x)y' + q(x)y = f(x)$

\* This also works for higher order equations.

<ul style="list-style-type: none"><li>Possible analogy: Even and odd numbers ↗ ↘ homog solutions    non homog. solutions</li></ul>	<p>Even + Even = Even Odd + Even = Odd Odd + Odd ≠ Odd</p> <p><b>Odd + Odd = even</b></p> <p>This is where the analogy breaks down So don't take it too seriously</p>
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Proof

$$\begin{aligned}
 & (y_c + y_p)'' + p(x)(y_c + y_p)' + q(x)(y_c + y_p) \\
 &= y_c'' + y_p'' + p(x)y_c' + p(x)y_p' + q(x)y_c + q(x)y_p \\
 &= \left[ y_c'' + p(x)y_c' + q(x)y_c \right] + \left[ y_p'' + p(x)y_p' + q(x)y_p \right] \\
 &= 0 + f(x) \\
 &= f(x) \qquad \text{Q.E.D.}
 \end{aligned}$$

General solution Consider  $n$ th order linear nonhomogeneous DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y = \underline{\underline{f(x)}}$$

If  $y_p$  is any particular solution of this equation, and  $y_1, y_2, \dots, y_n$  are  $n$  linearly independent solutions of the homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \dots + p_n(x)y = \underline{\underline{0}}$$

$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)$$

is the general solution of the nonhomogeneous equation.