

Constant coefficient linear homogeneous equations

This is an equation like

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

where $a_n, a_{n-1}, \dots, a_1, a_0$ are constants.

The basic idea is to try a solution of the form e^{rx}

$$y = e^{rx} \quad y' = r e^{rx} \quad y'' = r^2 e^{rx}, \dots, \quad y^{(n)} = r^n e^{rx}$$

Then
becomes

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

$$a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \dots + a_1 r e^{rx} + a_0 e^{rx} = 0$$

$$(a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0) e^{rx} = 0$$

Thus e^{rx} will be a solution if

$$\boxed{a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0} \quad \text{Characteristic Equation.}$$

To get Characteristic equation from the original homogeneous linear differential equation, replace
 $y \rightarrow 1$, $y' \rightarrow r$, $y'' \rightarrow r^2$, etc $y^{(n)} \rightarrow r^n$

To summarize: If r solves characteristic equation,
then $y = e^{rx}$ solves the differential equation.

Is this all there is to it?

If the solutions of the characteristic equation are real and distinct, then yes.

Let $n =$ order of differential equation $=$ degree of characteristic equation.

A degree n polynomial equation such as

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0$$

has n solutions r_1, \dots, r_n which may be complex ($a+bi$) and which may be repeated such that the polynomial factors as

$$a_n (r - r_1)(r - r_2)(r - r_3) \dots (r - r_n)$$

Theorem

If the characteristic equation has n solutions r_1, r_2, \dots, r_n that are all distinct and real, then the general solution is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$$

Fact: for r_1, r_2, \dots, r_n distinct, the functions $e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}$ are linearly independent

[Follows from the relation between the discriminant and the Vandermonde determinant, if you care.]

Example $y^{(3)} - 3y'' + 2y' = 0$

Characteristic equation $r^3 - 3r^2 + 2r = 0$

$$r(r^2 - 3r + 2) = 0$$

$$r(r-1)(r-2) = 0$$

So roots $r_1 = 0$ $r_2 = 1$ $r_3 = 2$ Real, distinct

general solution $y(x) = c_1 e^{0x} + c_2 e^x + c_3 e^{2x} = c_1 + c_2 e^x + c_3 e^{2x}$

What if a root is repeated?

Eg $y'' - 2y' + y = 0$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0 \quad \leftarrow \text{double solution } r_1 = 1 \quad r_2 = 1$$

So $y = e^{rx} = e^x$ is a solution

The general solution is NOT $y(x) = c_1 e^x + c_2 e^x$

because e^x and e^x are not linearly independent!

We need ANOTHER solution that CANNOT be found using the characteristic equation.

This requires some work, so we are going to introduce some theory that will make it more comprehensible.