

Constant coefficient differential operators.

What is $\frac{d}{dx}$? Well $\frac{d}{dx}[f(x)] = f'(x)$ derivative

The symbol " $\frac{d}{dx}$ " represents the operation of differentiation
the operation that takes a function and produces its derivative

$\frac{d}{dx}$ is therefore called the "Derivative operator"

Similarly $\frac{d^2}{dx^2}$ is an operator called the "second derivative operator"

And so on $\frac{d^2}{dx^3}, \dots, \frac{d^n}{dx^n} = n$ th derivative operator.

Now we can define algebraic operations on operators themselves

If A is an operator, then A^2 does the operation twice

So if $D = \frac{d}{dx}$ then $D^2 =$ take derivative twice $= \frac{d^2}{dx^2}$

and $D^3 = \frac{d^3}{dx^3}, \dots, D^n = \frac{d^n}{dx^n}$

We can also combine differential operators by addition

$$L = D^2 - 2D + \textcircled{3} \leftarrow \text{this is the operator that multiplies by 3}$$
$$= \frac{d^2}{dx^2} - 2\frac{d}{dx} + 3$$

Then $L[f(x)] = f''(x) - 2f'(x) + 3f(x)$

A general constant coefficient linear differential operator looks like

$$L = a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0$$

$$\begin{aligned} L \cdot y &= (a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0) \cdot y \\ &= a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y \end{aligned}$$

So this is a new notation for the LHS's of the differential equations we want to consider.

Observe the similarity between

Differential operator $L = a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0$

Characteristic polynomial $p(r) = a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0$

So we can write $L = p(D)$ if we think of D as substituted for r .

Just as we can factor polynomials, we can factor differential operators.

E.g. Factor $D^2 - 2D + 1 = (D - 1)^2$

$$\begin{array}{l} D^2 + 5D + 1 \\ r^2 + 5r + 1 \end{array} \quad \frac{-5 \pm \sqrt{25 - 4}}{2} = \frac{-5 \pm \sqrt{21}}{2}$$

$$D^2 + 5D + 1 = \left[D - \left(\frac{-5 + \sqrt{21}}{2} \right) \right] \left[D - \left(\frac{-5 - \sqrt{21}}{2} \right) \right]$$

Key fact: For constant coefficient differential operators, the order of the factors does not matter.

$$(D-a)(D-b) = D^2 - (a+b)D + ab = (D-b)(D-a)$$

[This isn't true for nonconstant coefficients like $(D-x)$]

Lemma If $L = f(D)g(D)$ for polynomials $f(r)$ and $g(r)$

Then any function y_1 satisfying $f(D)y = 0$ also satisfies $Ly = 0$
and any function y_2 satisfying $g(D)y = 0$ also satisfies $Ly = 0$

Proof $Ly_1 = f(D)g(D)y_1 = g(D)f(D)y_1$ b/c order doesn't matter here
 $= g(D)[f(D)y_1] = g(D)[0] = 0$

And $Ly_2 = f(D)g(D)y_2 = f(D)[g(D)y_2] = f(D)[0] = 0$.

This is useful b/c it says we can solve the two equations $f(D)y = 0$ and $g(D)y = 0$ and get solutions to $Ly = 0$.

- Case of repeated roots. The "worst case" would be just one root of the characteristic equation

$$p(r) = (r-a)^n = 0 \quad \text{ie} \quad L = p(D) = (D-a)^n$$

Q: What are solutions of $(D-a)^n y = 0$?

We know $y = e^{ax}$ works, since $(D-a)e^{ax} = 0$