

Solving Nonhomogeneous equations I

Undetermined coefficients

This technique grows out of the "trial-and-error method"

Find a particular solution of $y'' + y' + 2y = \sin(x)$

$$p(D) = D^2 + D + 2 \quad \text{Want to solve } p(D)y = \sin(x)$$

Why not try a multiple of $\sin(x)$, $A \sin(x)$?

$$\begin{aligned} p(D)(A \sin(x)) &= A \sin''(x) + A \sin'(x) + 2A \sin(x) \\ &= -A \sin(x) + A \cos(x) + 2A \sin(x) \\ &= A \sin(x) + A \cos(x) \end{aligned}$$

If we take $A=1$, we get $\sin(x)$, but also $\cos(x)$.
So why not include a $\cos(x)$ term as well?

$$\begin{aligned} p(D)(A \sin(x) + B \cos(x)) &= (D^2 + D + 2)(A \sin(x) + B \cos(x)) \\ &= -A \sin(x) - B \cos(x) + A \cos(x) - B \sin(x) + 2A \sin(x) + 2B \cos(x) \\ &= (-A - B + 2A) \sin(x) + (-B + A + 2B) \cos(x) \\ &= (A - B) \sin(x) + (A + B) \cos(x) \end{aligned}$$

Want this = $\sin(x)$, so want $A - B = 1$ $A = \frac{1}{2}$
 $A + B = 0$ $B = -\frac{1}{2}$

Thus $y_p = \frac{1}{2} \sin(x) - \frac{1}{2} \cos(x)$ is a particular solution.

The general solution is gotten by adding to this the general solution of the homogeneous equation $(D^2 + D + 2)y = 0$

$$r^2 + r + 2 = 0 \quad r = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm i\sqrt{7}}{2}$$

$$\text{Solutions } y_c = e^{-\frac{1}{2}x} \left(C_1 \cos\left(\frac{\sqrt{7}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{7}}{2}x\right) \right)$$

General solution of $(D^2 + D + 2)y = \sin(x)$ is

$$y = \underbrace{e^{-\frac{1}{2}x} \left(C_1 \cos\left(\frac{\sqrt{7}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{7}}{2}x\right) \right)}_{y_c} + \underbrace{\frac{1}{2} \sin(x) - \frac{1}{2} \cos(x)}_{y_p}$$

$$p(D)(y_c + y_p) = p(D)y_c + p(D)y_p = 0 + \sin(x) \quad \checkmark$$

We can make this "trial and error" method more systematic, but the question is "What am I supposed to try?"

Somewhat imprecise but useful suggestion:

"Try a linear combination of all the terms that appear in the derivatives of the nonhomogeneous term."

Let's consider $p(D) = D^2 + D + 2$ again, but now we want to solve $p(D)y = x^3$

Nonhomogeneous term	$= x^3$	} So try $Ax^3 + Bx^2 + Cx + E$
first derivative	$3x^2$	
second "	$6x$	
third "	6	
fourth "	0	

$$\begin{aligned}
 & (D^2 + D + 2)(Ax^3 + Bx^2 + Cx + E) \\
 &= 6Ax + 2B + 3Ax^2 + 2Bx + C + 2Ax^3 + 2Bx^2 + 2Cx + 2E \\
 &= 2Ax^3 + (3A + 2B)x^2 + (6A + 2B + 2C)x + (2B + C + 2E)
 \end{aligned}$$

$2A = 1$	$A = \frac{1}{2}$	
$3A + 2B = 0$	$\frac{3}{2} + 2B = 0$	$B = -\frac{3}{4}$
$6A + 2B + 2C = 0$	$3 - \frac{3}{2} + 2C = 0$	$\frac{3}{2} + 2C = 0$ $C = -\frac{3}{4}$
$2B + C + 2E = 0$	$-\frac{3}{2} - \frac{3}{4} + 2E = 0$	$-\frac{9}{4} + 2E = 0$ $E = \frac{9}{8}$

$y_p = \frac{1}{2}x^3 - \frac{3}{4}x^2 - \frac{3}{4}x + \frac{9}{8}$

→ a particular solution

How about $(D^2 + D + 2)y = e^{4x}$

Nonhomogeneous term	$= e^{4x}$	} So just try $y_p = Ae^{4x}$
First derivative	$= 4e^{4x}$	
Second "	$= 16e^{4x}$	
Third "	$= 64e^{4x}$	

$$(D^2 + D + 2)Ae^{4x} = 16Ae^{4x} + 4Ae^{4x} + 2Ae^{4x} = 22Ae^{4x}$$

Want $22A = 1$ so $A = \frac{1}{22}$

$y_p = \frac{1}{22}e^{4x}$