Nonhomogeneous equations II – The resonant case

Sometimes the method of undetermined coefficients will fail.

For example, Try solving $(D-2)y = e^{2x}$ or $y' - 2y = e^{2x}$

Based on what we said last time, we should try Ae^{2x}

$$(D-2)(Ae^{2x}) = 2Ae^{2x} - 2Ae^{2x} = 0 \overset{?}{=} e^{2x}$$

It can't work!

The problem is that the nonhomogeneous term e^{2x} has something in common with the solutions of the homogeneous equation $(D-2)y = 0$!

This is called (mathematical) <u>resonance</u>.

Another example: $(D^2+D)y = x$

$y = Ax + B$ doesn't work, but $y = Ax^2 + Bx + C$ does

$$(D^2+D)(Ax^2+Bx+C) = 2A + 2Ax + B$$
$$= 2Ax + (2A+B)$$

want $= x$

So $2A = 1$ $\quad A = 1/2$ $\quad y_p = \frac{1}{2}x^2 - x$

$2A + B = 0$ $\quad B = -1$

$C =$ anything!

General slogan: Need to include higher power of x times the functions you already have

Another example $(D^2+1)y = \sin(x)$

Since $\sin(x)$ and $\cos(x)$ both solve $(D^2+1)y=0$, we need to include $x\sin(x)$ and $x\cos(x)$

$$(D^2+1)\big(Ax\sin(x) + Bx\cos(x) + C\sin(x) + D\cos(x)\big)$$

$$D^2(x\sin x) = D(x\cos x + \sin x) = -x\sin x + \cos x + \cos x$$
$$(D^2+1)(x\sin x) = 2\cos x$$
$$D^2(x\cos x) = D(-x\sin x + \cos x) = -x\cos x - \sin x - \sin x$$
$$(D^2+1)(x\cos x) = -2\sin x$$

$$A\cdot 2\cos x - B2\sin x + 0 + 0$$

want $= \sin x$

So let $A=0$, $B=-\frac{1}{2}$

$$\boxed{y_p = -\tfrac{1}{2}x\cos(x)}$$

Why does this work? How high of a power of x must you take? Answers in terms of a more systematic method of solving nonhomogeneous equations.

The Annihilator method

Consider a function $f(x)$. An annihilator of $f(x)$ is a differential operator $A(D)$ such that

$$A(D)(f(x)) = 0$$

$f(x)$	$A(D)$
x^3	D^4
e^{ax}	$D-a$
x^2e^{ax}	$(D-a)^3$
$\sin(3x)$	D^2+9
$x\sin(3x)$	$(D^2+9)^2$
$e^{ax}+e^{bx}$	$(D-a)(D-b)$
$xe^{ax}+x^2e^{bx}$	$(D-a)^2(D-b)^3$
$\vdots$	$\vdots$

To solve
$$p(D)y = f(x)$$
Find annihilator $A(D)$ so that $A(D)\cdot f(x) = 0$ (if possible)

Then apply $A(D)$ to both sides
$$p(D)y = f(x)$$
$$A(D)\,p(D)\,y = A(D)\,f(x) = 0$$

So the function y we want also solves the homogeneous equation
$$A(D)\,p(D)\,y = 0.$$

Find the general solution of this equation, and try <u>that</u> in the nonhomogeneous equation.

<u>Example</u> $(D-5)y = xe^{5x}$

$f(x) = xe^{5x}$ annihilator $= (D-5)^2$

$$(D-5)^2(D-5)y = (D-5)^2(xe^{5x}) = 0$$
$$(D-5)^3y = 0$$

So $y = Ae^{5x} + Bxe^{5x} + Cx^2e^{5x}$