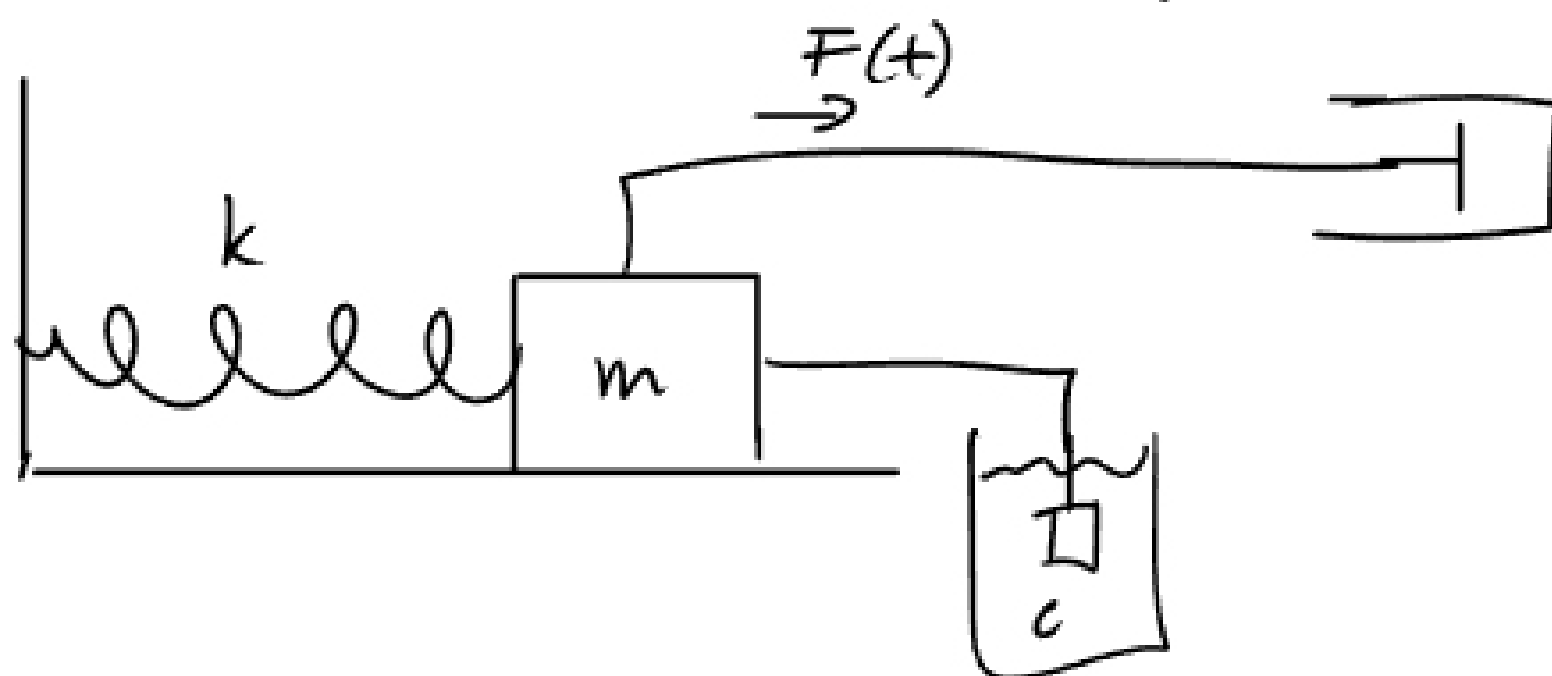


Forced oscillations I

We return to the damped oscillator, but this time we install an external driving force.



Thus Newton's 2nd law gives $ma = -kx - cv + F(t)$

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = F(t)$$

Since the system is subject to an external force, we get a Nonhomogeneous equation.

We will consider a periodic driving force of the form

$$F(t) = F_0 \cos(\omega t)$$

or

$$F(t) = F_0 \sin(\omega t)$$

ω = angular frequency (radians/second)

$\frac{\omega}{2\pi}$ = ordinary frequency (cycles/second = Hertz = Hz)

$T = \frac{2\pi}{\omega}$ = period (seconds/cycle)

remember 1 cycle = 2π radians.

Undamped case, $c = 0$:

$$m x'' + kx = F_0 \cos \omega t$$

Try undetermined coefficients

$$\left. \begin{array}{l} \text{Non homog. term } F_0 \cos \omega t \\ \text{1st deriv } -\omega F_0 \sin \omega t \\ \text{2nd deriv } -\omega^2 F_0 \cos \omega t \end{array} \right\} \begin{array}{l} \text{just get } \sin \omega t \\ \text{ \& } \cos \omega t \end{array}$$

So try $x(t) = A \sin \omega t + B \cos \omega t$

$$\begin{aligned} m x'' + kx &= -m\omega^2 A \sin \omega t - m\omega^2 B \cos \omega t + kA \sin \omega t + kB \cos \omega t \\ &= (-m\omega^2 + k) A \sin \omega t + (-m\omega^2 + k) B \cos \omega t \end{aligned}$$

Want this = $F_0 \cos \omega t$

Thus, $A = 0$ and $(-m\omega^2 + k) B = F_0$

so $B = \frac{F_0}{k - m\omega^2}$ $x_P(t) = \frac{F_0}{k - m\omega^2} \cos \omega t$

Recall notation $\omega_0 = \sqrt{\frac{k}{m}}$ natural frequency.

$$\frac{F_0 / m}{k - m\omega^2 / m} = \frac{F/m}{\frac{k}{m} - \omega^2} = \frac{F/m}{\omega_0^2 - \omega^2}$$

So the amplitude of the particular solution depends on the difference between the driving frequency and the natural frequency.

What happens if $\omega = \omega_0$? Driving freq = natural freq.
 $k = m\omega^2$

Then we would be dividing by zero! It doesn't make sense! This is (physical) resonance.

And look if $k = m\omega^2$, $(mD^2 + k)(F_0 \cos \omega t)$
 $= (-m\omega^2 + k)F_0 \cos \omega t = 0$

So driving force solves the homogeneous equation that is mathematical resonance that we talked about last time.

Resonant case: To solve $mx'' + kx = F_0 \cos(\omega_0 t)$
where $\omega_0 = \sqrt{\frac{k}{m}}$, we try

$$x(t) = At \sin \omega_0 t + Bt \cos \omega_0 t$$

$$x'(t) = A(\omega_0 t \cos \omega_0 t + \sin \omega_0 t) + B(-\omega_0 t \sin \omega_0 t + \cos \omega_0 t)$$

$$x''(t) = A(-\omega_0^2 t \sin \omega_0 t + \omega_0 \cos \omega_0 t + \omega_0 \cos \omega_0 t) \\ + B(-\omega_0^2 t \cos \omega_0 t - \omega_0 \sin \omega_0 t - \omega_0 \sin \omega_0 t)$$

$$m x''(t) + k x(t) = m(x'' + \omega_0^2 x)$$

$$= m(A \cdot 2\omega_0 \cos \omega_0 t - B \cdot 2\omega_0 \sin \omega_0 t)$$

$$\text{Want } = F_0 \cos \omega_0 t$$

$$\text{So } B = 0 \text{ and } 2Am\omega_0 = F_0$$

$$x_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

