

Forced oscillations II

Damped case: $mx'' + cx' + kx = F_0 \cos \omega t$

Undetermined coefficients suggests to try

$$x(t) = A \cos \omega t + B \sin \omega t$$

$$x'(t) = -\omega A \sin \omega t + \omega B \cos \omega t$$

$$x''(t) = -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t$$

$$mx'' + cx' + kx = \left[(k - m\omega^2)A + c\omega B \right] \cos \omega t \\ + \left[(k - m\omega^2)B - c\omega A \right] \sin \omega t$$

$$\Sigma \quad \begin{aligned} (k - m\omega^2)A + c\omega B &= F_0 \\ -c\omega A + (k - m\omega^2)B &= 0 \end{aligned}$$

$$B = \frac{c\omega}{k - m\omega^2} A$$

$$(k - m\omega^2)A + \frac{(c\omega)^2}{(k - m\omega^2)} A = F_0$$

$$\left[(k - m\omega^2)^2 + (c\omega)^2 \right] A = (k - m\omega^2) F_0$$

$$A = \frac{(k - m\omega^2) F_0}{(k - m\omega^2)^2 + (c\omega)^2} \Rightarrow B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}$$

We can write $A \cos \omega t + B \sin \omega t$
as $C \cos(\omega t - \alpha)$

$$\text{where } C = \sqrt{A^2 + B^2} \quad \tan \alpha = \frac{B}{A}$$

Most important is $C = \text{amplitude}$

$$C^2 = A^2 + B^2 = \frac{(k - m\omega^2)^2 F_0^2 + (c\omega)^2 F_0^2}{((k - m\omega^2)^2 + (c\omega)^2)^2}$$
$$= \frac{F_0^2}{(k - m\omega^2)^2 + (c\omega)^2}$$

$$C = \text{Amplitude} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

When the damping c is not zero, the denominator is always positive.

$$c \neq 0 \Rightarrow (c\omega)^2 > 0 \Rightarrow (k - m\omega^2)^2 + (c\omega)^2 > 0$$

So strictly speaking, there is no "resonance" in the damped system.

But we can still plot C vs. ω

$$\text{if } \omega \text{ very small, } C \approx \frac{F_0}{k}$$

$$\text{if } \omega \text{ very large } C \approx 0$$

When does $C = \frac{F_0}{\sqrt{(k-m\omega^2)^2 + (c\omega)^2}}$ have a maximum?

Need $\sqrt{(k-m\omega^2)^2 + (c\omega)^2}$ has a minimum

Need $(k-m\omega^2)^2 + (c\omega)^2$ has a minimum

$$0 = \frac{d}{d\omega} [(k-m\omega^2)^2 + (c\omega)^2] = 2(k-m\omega^2)(-2m\omega) + 2c^2\omega$$

$\omega = 0$ is a solution, we want another one, so cancel ω

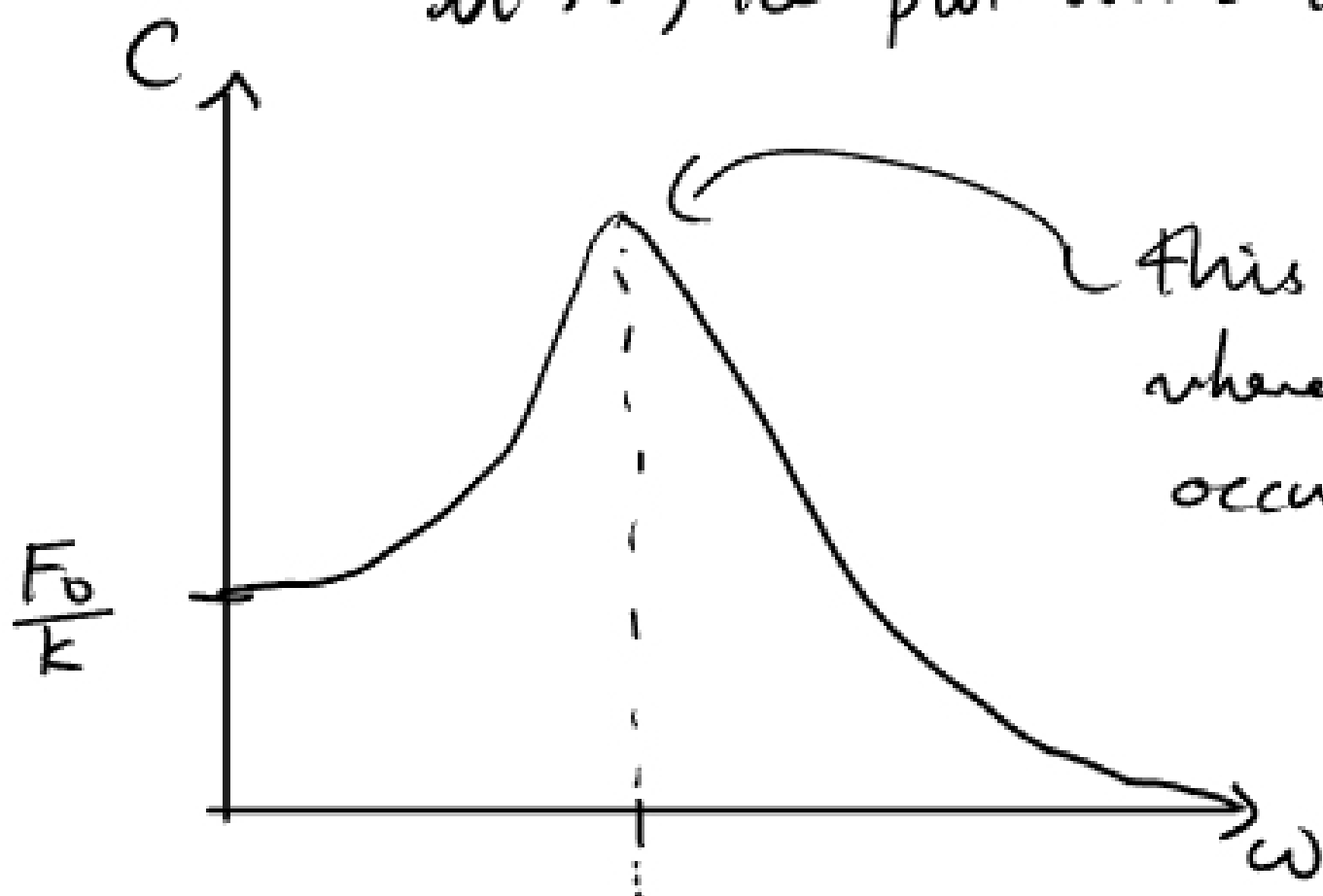
$$0 = -4m(k-m\omega^2) + 2c^2 \quad \rightarrow \quad k - \frac{c^2}{2m} = m\omega^2$$

$$0 = -2m(k-m\omega^2) + c^2 \quad \rightarrow \quad \frac{k}{m} - \frac{c^2}{2m^2} = \omega^2$$
$$(k-m\omega^2) = \frac{c^2}{2m}$$

$$\omega = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}}$$
$$= \sqrt{\omega_0^2 - 2\gamma^2}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$
$$\gamma = \frac{c}{2m}$$

This frequency may not be real, but if it is, the plot looks like



This is the frequency where "practical resonance" occurs.