Fourier Coefficients:

Recall orthogonality relations

Fundamental period 2π

$\sin nt$, $\cos nt$, $n=1,2,3,\dots$

$$\int_{-\pi}^{\pi} \cos mt \cos nt \, dt = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mt \sin nt \, dt = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mt \cos nt \, dt = 0$$

Fundamental period $2L$

$\sin \frac{n\pi t}{L}$, $\cos \frac{n\pi t}{L}$, $n=1,2,3,\dots$

$$\int_{-L}^{L} \cos \frac{m\pi t}{L} \cos \frac{n\pi t}{L} \, dt = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

$$\int_{-L}^{L} \sin \frac{m\pi t}{L} \sin \frac{n\pi t}{L} \, dt = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin \frac{m\pi t}{L} \cos \frac{n\pi t}{L} \, dt = 0$$

Now, suppose we are trying to write a function $f(t)$ (which we assume is periodic with period $2L$) as a sum of sines and cosines with fundamental period $2L$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{L} + b_n \sin \frac{n\pi t}{L} \right)$$

To figure out constant term, just integrate from $-L$ to L

$$\int_{L}^{L} f(t)\,dt = \int_{-L}^{L}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right) dt$$

look: $\int_{-L}^{L} \cos\frac{n\pi t}{L}\,dt = \left[\frac{L}{\pi n}\sin\frac{n\pi t}{L}\right]_{-L}^{L} = \frac{L}{\pi n}\left[\sin n\pi - \sin -n\pi\right]$

$$= \frac{L}{\pi n}\cdot 0 = 0$$

$$\int_{-L}^{L} \sin\frac{n\pi t}{L}\,dt = \left[\frac{-L}{\pi n}\cos\frac{n\pi t}{L}\right]_{-L}^{L} = -\frac{L}{\pi n}\left[\cos n\pi - \cos(-n\pi)\right]$$

$$= \frac{-L}{\pi n}\cdot 0 = 0$$

So all of the integral goes away. except for $\frac{a_0}{2}$ term

$$\int_{-L}^{L} f(t)\,dt = \int_{-L}^{L}\frac{a_0}{2}\,dt = \frac{a_0}{2}\int_{-L}^{L} dt = \frac{a_0}{2}2L = a_0 L$$

$$\int_{-L}^{L} f(t)\,dt = a_0 L \implies a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt$$

So we found a_0 in terms of $f(t)$!

To find a_m: multiply by $\cos\frac{m\pi t}{L}$ and integrate:
use orthogonality

$$\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt = \int_{-L}^{L}\left(\frac{a_0}{2} + \sum a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L}\right)\cos\frac{m\pi t}{L}\,dt$$

$$= \int_{-L}^{L} \frac{a_0}{2}\cos\frac{m\pi t}{L} + \sum \quad a_n \cos\frac{n\pi t}{L}\cos\frac{m\pi t}{L} + b_n \sin\frac{n\pi t}{L}\cos\frac{m\pi t}{L}\Big)\,dt$$

By orthogonality, all the terms go away except

$$\int_{-L}^{L} a_n \cos\frac{n\pi t}{L} \cos\frac{m\pi t}{L}\,dt \quad \text{when } m=n$$

$$= a_m L$$

So $\int_{-L}^{L} f(t)\cos\frac{m\pi t}{L}\,dt$ so $a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{m\pi t}{L}\,dt$

Similar argument shows $b_m = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{m\pi t}{L}\,dt$

Definition The Fourier coefficients of $f(t)$ are

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt \quad n=1,2,\ldots$$

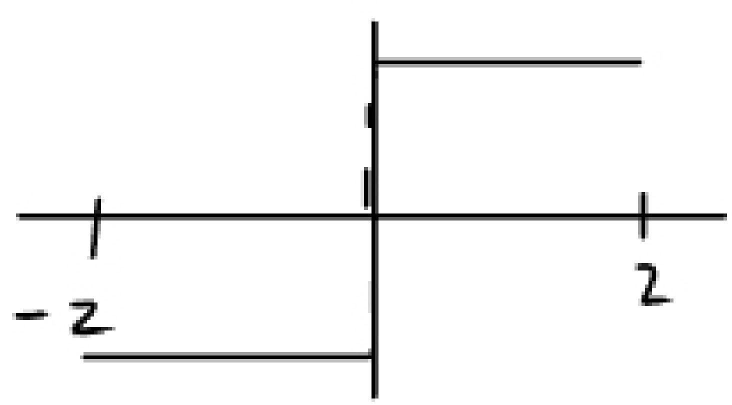
$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt \quad n=1,2,\ldots$$

The Fourier Series of $f(t)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi t}{L} + b_n\sin\frac{n\pi t}{L}\right)$$

Examples: square-wave with period $4 = 2L$, $L = 2$.

$$f(t) = \begin{cases} 1 & 0 \le t < 2 \\ -1 & -2 \le t < 0 \end{cases}$$



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