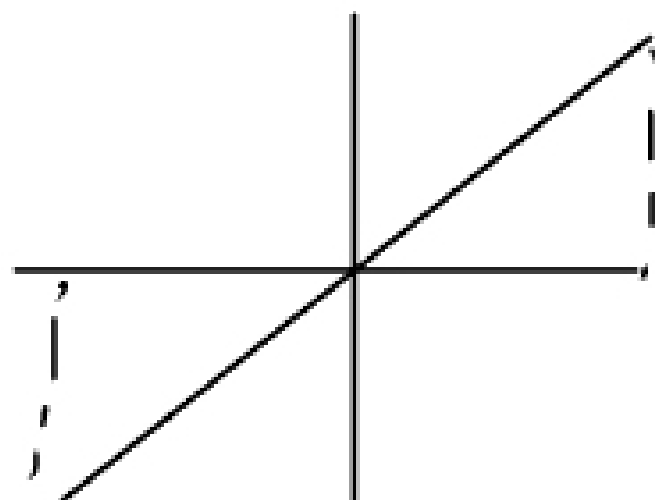
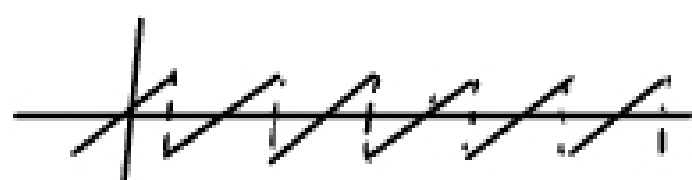


Fourier Series; Pointwise convergence

Another example of Fourier Series: Sawtooth wave



$$f(t) = \begin{cases} t & \text{if } -1 \leq t < 1, \\ \text{repeats periodically with} \\ \text{period } 2 \quad (L=1). \end{cases}$$



What is $f(27.2)$?

$$f(27.2) = f(25.2) = f(23.2) = \dots = f(1.2) = f(-.8) = -.8$$

In general, for a function with period $2L$: $f(t+2Lk) = f(t)$

Compute Fourier coefficients

$$a_0 = \frac{1}{1} \int_{-1}^1 f(t) dt = \int_{-1}^1 t dt = \left[\frac{1}{2} t^2 \right]_{-1}^1 = \frac{1}{2} (1 - 1) = 0$$

$$a_n = \frac{1}{1} \int_{-1}^1 f(t) \cos \frac{n\pi t}{1} dt = \int_{-1}^1 t \cos n\pi t dt$$

Parts $u = t$ $dv = \cos n\pi t dt$
 $du = dt$ $v = \frac{1}{n\pi} \sin n\pi t$

$$\begin{aligned} \int t \cos n\pi t dt &= \frac{1}{n\pi} t \sin n\pi t - \int \frac{1}{n\pi} \sin n\pi t dt \\ &= \frac{1}{n\pi} t \sin n\pi t + \left(\frac{1}{n\pi} \right)^2 \cos n\pi t + C \end{aligned}$$

$$\begin{aligned}
 a_n &= \left[\frac{1}{n\pi} t \sin n\pi t + \left(\frac{1}{n\pi}\right)^2 \cos n\pi t \right]_{-1}^1 \\
 &= \frac{1}{n\pi} \sin n\pi + \frac{1}{n\pi} \sin(-n\pi) + \left(\frac{1}{n\pi}\right)^2 \cos n\pi - \left(\frac{1}{n\pi}\right)^2 \cos(-n\pi) \\
 &= 0 + 0 + \left(\frac{1}{n\pi}\right)^2 - \left(\frac{1}{n\pi}\right)^2 = 0
 \end{aligned}$$

So $a_0 = 0$ and $a_n = 0$.

We could have seen this because $f(t)$, $f(t)\cos n\pi t$ are odd functions.

• If $f(t)$ is odd: $f(-t) = -f(t)$ Then $\int_{-a}^a f(t) dt = 0$

$$b_n = \frac{1}{1} \int_{-1}^1 f(t) \sin n\pi t dt = \int_{-1}^1 t \sin n\pi t dt$$

Parts: $\int t \sin n\pi t dt = -\frac{1}{n\pi} t \cos n\pi t - \int -\frac{1}{n\pi} \cos n\pi t dt$

$$\begin{array}{l}
 u = t \quad dv = \sin n\pi t dt \\
 du = dt \quad v = -\frac{1}{n\pi} \cos n\pi t
 \end{array}$$

$$= -\frac{1}{n\pi} t \cos n\pi t + \left(\frac{1}{n\pi}\right)^2 \sin n\pi t + C$$

$$b_n = \left[-\frac{1}{n\pi} t \cos n\pi t + \left(\frac{1}{n\pi}\right)^2 \sin n\pi t \right]_{-1}^1$$

$$= -\frac{1}{n\pi} \cos n\pi + \frac{1}{n\pi} (-1) \cos(-n\pi) + \left(\frac{1}{n\pi}\right)^2 \sin n\pi - \left(\frac{1}{n\pi}\right)^2 \sin(-n\pi)$$

$$= -\frac{1}{n\pi} (\cos n\pi + \cos n\pi) + 0 - 0$$

$$= -\frac{2}{n\pi} \cos n\pi = \begin{cases} \frac{2}{n\pi} & n \text{ odd} \\ -\frac{2}{n\pi} & n \text{ even} \end{cases}$$

We can also write $\cos n\pi = (-1)^n$

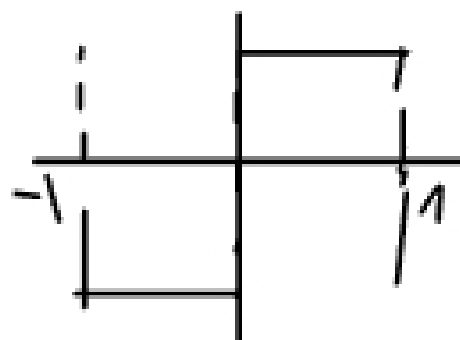
$$\text{so } b_n = -\frac{2}{n\pi} (-1)^n = \frac{2}{n\pi} (-1)^{n+1}$$

So the Fourier series for the sawtooth is

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^{n+1} \sin n\pi t$$

Compare with square wave of same period

$$\sum_{n \text{ odd}} \frac{4}{n\pi} \sin n\pi t$$



In the square wave, only odd multiples of the fundamental occur
 In the sawtooth, all multiples of the fundamental occur.
 In both, the amplitude of the n th multiple is proportional to $1/n$.

For triangle wave



Odd multiples of fundamental are present, but the n th has amplitude proportional to $1/n^2$.