Forced Oscillation and Fourier Series.

Recall Forced oscillator: $mx'' + cx' + kx = F(t)$

Today, we only consider undamped case $c=0$

$$mx'' + kx = F(t).$$

We will solve this equation for nonhomogeneous terms of increasing complexity.

(Case 0) $\underline{F(t)=0}$: $mx'' + kx = 0$

characteristic equation $mr^2 + k = 0$

$$r = \pm\sqrt{\frac{-k}{m}} = \pm i\sqrt{\frac{k}{m}}$$

general solution $x(t) = c_1 \cos\sqrt{\frac{k}{m}}\,t + c_2 \sin\sqrt{\frac{k}{m}}\,t$

This reveals the physical meaning of $\sqrt{\frac{k}{m}}$, it is the

Natural angular frequency: $\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$

(Case 1) $\underline{F(t) = F_0 \cos\omega t}$: We use undetermined coefficients to find a periodic solution.

Try $x(t) = A\cos\omega t$

$$mx'' + kx = -m\omega^2 A\cos\omega t + kA\cos\omega t$$
$$= (k - m\omega^2) A\cos\omega t$$

We want this to equal $F(t) = F_0\cos\omega t$,

So we need

$$(k - m\omega^2)A = F_0$$

$$A = \frac{F_0}{k - m\omega^2} \to x(t) = \frac{F_0}{k - m\omega^2}\cos\omega t$$

We can write $\dfrac{F_0}{k-m\omega^2} = \dfrac{F_0/m}{(k-m\omega^2)/m} = \dfrac{F_0/m}{\frac{k}{m}-\omega^2} = \dfrac{F_0/m}{\omega_0^2-\omega^2}$

In summary, a periodic solution of $mx''+kx = F_0\cos\omega t$ is

$$x(t) = \frac{F_0/m}{\omega_0^2-\omega^2}\cos\omega t$$

This only works if $\omega \neq \omega_0$, that is, the driving frequency is not equal to the natural frequency.
If, conversely, $\omega = \omega_0$, resonance occurs, and there is no periodic solution.

(Case 2) $F(t) = F_1\cos\omega_1 t + F_2\cos\omega_2 t$

For this case we will use the previous case plus a version of the principle of superposition:

Suppose $x_1(t)$ satisfies $mx_1''+kx_1 = F_1(t)$
and $x_2(t)$ satisfies $mx_2''+kx_2 = F_2(t)$
Then $x(t) = x_1(t)+x_2(t)$ satisfies

$$mx''+kx = F_1(t)+F_2(t)$$

Proof:
$$\begin{aligned} mx''+kx &= m(x_1+x_2)''+k(x_1+x_2) \\ &= mx_1''+mx_2''+kx_1+kx_2 \\ &= \underbrace{mx_1''+kx_1}_{F_1(t)} + \underbrace{mx_2''+kx_2}_{F_2(t)} \\ &= F_1(t)+F_2(t) \end{aligned}$$

This holds more generally for any linear differential equation.

suppose $\begin{Bmatrix} p(D)\, y_1 = f_1 \\ p(D)\, y_2 = f_2 \end{Bmatrix}$ then $p(D)\,(y_1+y_2) = p(D)\,y_1 + p(D)\,y_2 = f_1+f_2$

So, to solve $mx'' + kx = F_1 \cos\omega_1 t + F_2 \cos\omega_2 t$

Solve $mx_1'' + kx_1 = F_1 \cos\omega_1 t$
$mx_2'' + kx_2 = F_2 \cos\omega_2 t$

$$x_1(t) = \frac{F_1/m}{\omega_0^2 - \omega_1^2} \cos\omega_1 t$$

$$x_2(t) = \frac{F_2/m}{\omega_0^2 - \omega_2^2} \cos\omega_2 t$$

So $x(t) = x_1(t) + x_2(t) = \dfrac{F_1/m}{\omega_0^2 - \omega_1^2} \cos\omega_1 t + \dfrac{F_2/m}{\omega_0^2 - \omega_2^2} \cos\omega_2 t$

is a particular solution of the original equation.

Eg. Solve $x'' + x = \cos 2t + \cos 3t$
Here, $m = 1$, $k = 1$, $\omega_0 = \sqrt{\frac{k}{m}} = 1$

Solve $x_1'' + x_1 = \cos 2t \Longrightarrow x_1(t) = \dfrac{1}{1^2 - 2^2} \cos 2t = \dfrac{-1}{3} \cos 2t$

$x_2'' + x_2 = \cos 3t \Longrightarrow x_2(t) = \dfrac{1}{1^2 - 3^2} \cos 3t = \dfrac{-1}{8} \cos 3t$

So $x(t) = \dfrac{-1}{3} \cos 2t - \dfrac{1}{8} \cos 3t$ solves original eqn.