Eigenvalues and Eigenfunctions

Last time: End point value problem

$$\left\{\begin{array}{l} y'' + \lambda y = 0 \\ y(0) = 0 \\ y(1) = 0 \end{array}\right\}$$

If $\lambda > 0$ there are two possibilities

- $\lambda = (n\pi)^2$ where n is an integer
  Then we have
  $y(x) = C \sin n\pi x$ as a solution
  for any constant C

- $\lambda \neq (n\pi)^2$, the only solution is $y(x) \equiv 0$.
  (y is constant equal to 0)

By convention, we declare the function $y(x) \equiv 0$ which is constantly equal to zero to be "<u>uninteresting</u>" or "<u>trivial</u>" or "<u>nugatory</u>".

All other functions are called "<u>interesting</u>" or "<u>nontrivial</u>."

So our problem only has interesting/nontrivial solutions if $\lambda = (n\pi)^2$ for some integer n.
The values of λ for which the problem has interesting/nontrivial solutions are called <u>eigenvalues</u>.

The interesting/nontrivial solutions themselves are called <u>eigenfunctions</u> or <u>modes</u>.

The set of eigenvalues is called the <u>spectrum</u> of the problem.

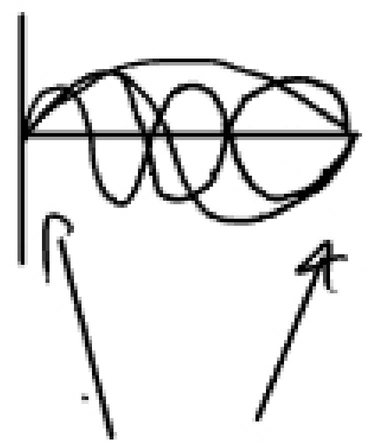
To summarize

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 \\ y(1) = 0 \end{cases}$$

The positive eigenvalues are
$\lambda = \pi^2, (2\pi)^2, (3\pi)^2, \ldots$
That is $\lambda = (n\pi)^2 \quad n = 1, 2, 3, \ldots$

Associated to the eigenvalue $\lambda_n = (n\pi)^2$, we have the basic eigenfunction:

$$y_n(x) = \sin n\pi x$$



the other eigenfunctions are $C y_n(x) = C \sin n\pi x$ where C is any constant.

What about $\lambda \leq 0$?
Is $\lambda = 0$ an eigenvalue?

$$\begin{cases} y'' = 0 \\ y(0) = 0 \\ y(1) = 0 \end{cases}$$

$y'' = 0 \Rightarrow y = c_1 + c_2 x$
$y(0) = 0 \Rightarrow 0 = c_1 \Rightarrow y = c_2 x$
$y(1) = 0 \Rightarrow 0 = c_2 \Rightarrow y \equiv 0 \Rightarrow y$ is trivial
So 0 is <u>not an eigenvalue</u>.

What about $\lambda < 0$?

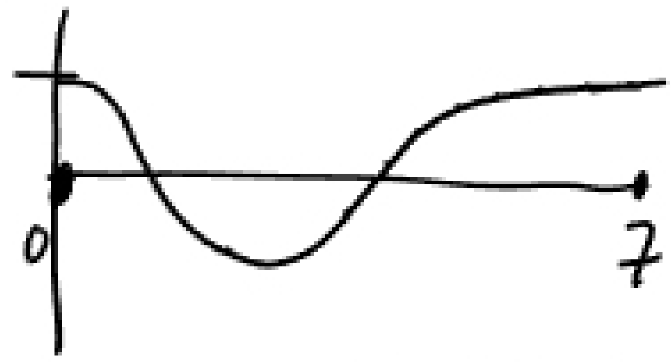
$y'' + \lambda y = 0$
$r^2 + \lambda = 0 \qquad r = \pm\sqrt{-\lambda}$
this is real

$y = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}$
$y(0) = 0 \Rightarrow c_1 + c_2 = 0$
$y(1) = 0 \Rightarrow c_1 e^{\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}} = 0 \Rightarrow c_1 e^{2\sqrt{\lambda}} + c_2 = 0$
$\Rightarrow c_1 e^{2\sqrt{\lambda}} - c_1 = 0 \Rightarrow c_1 (e^{2\sqrt{\lambda}} - 1) = 0 \Rightarrow c_1 = 0$
$\Rightarrow c_2 = -c_1 = 0$ so $y \equiv 0 \Rightarrow y$ is trivial
So $\lambda < 0$ is <u>not an eigenvalue</u>.

We can also study several variations on this problem. We'll keep the differential equation the same, but change the endpoint conditions:

Vanishing derivative at endpoints

$$\begin{cases} y''+\lambda y=0 & (1) \\ y'(0)=0 & (2) \\ y'(7)=0 & (3) \end{cases}$$



curve should have horizontal tangent at endpoints.

What are eigenvalues and eigenfunctions?
Consider $\lambda>0$.

(1) $y''+\lambda y=0 \Rightarrow y=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x$

$$y'=-c_1\sqrt{\lambda}\sin\sqrt{\lambda}x+c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

(2) $y'(0)=0 \Rightarrow -c_1\sqrt{\lambda}\cdot 0+c_2\sqrt{\lambda}\cdot 1=0$
$\Rightarrow c_2=0$

So $y=c_1\cos\sqrt{\lambda}x$

What about (3)? $y'(7)=-c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\cdot 7)=0$
This will force $c_1=0$ unless $\sin(\sqrt{\lambda}\cdot 7)=0$

$\sin(\sqrt{\lambda}\cdot 7)=0 \iff \sqrt{\lambda}\cdot 7=n\pi$ for some integer n
$\iff \lambda=\left(\frac{n\pi}{7}\right)^2$