

Heat Equation I

We consider a function $u(x, t)$. The independent variables are x (spatial position) and t (time). The dependent variable u represents temperature.

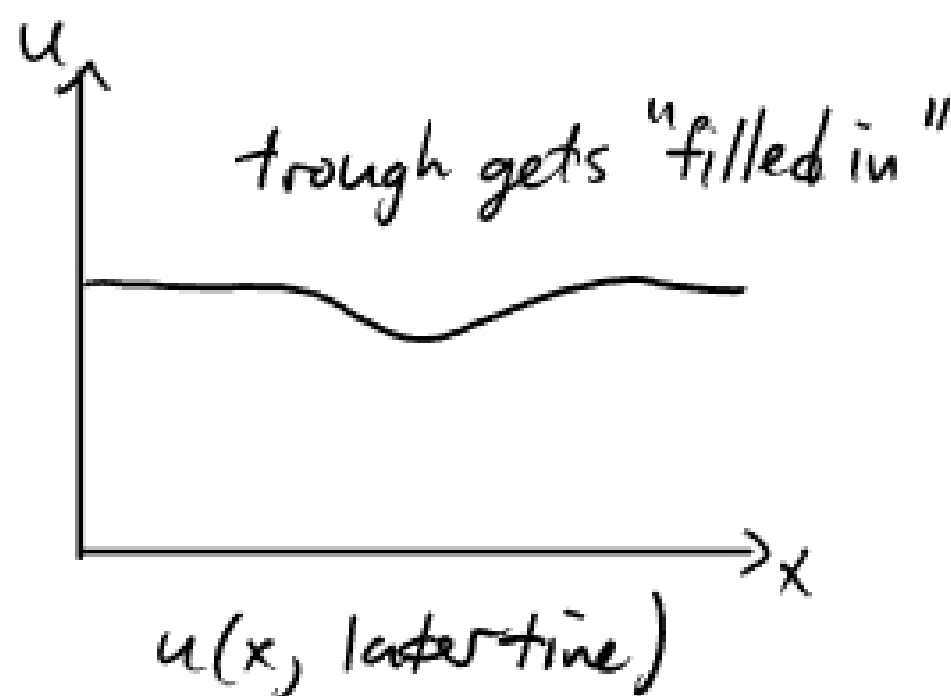
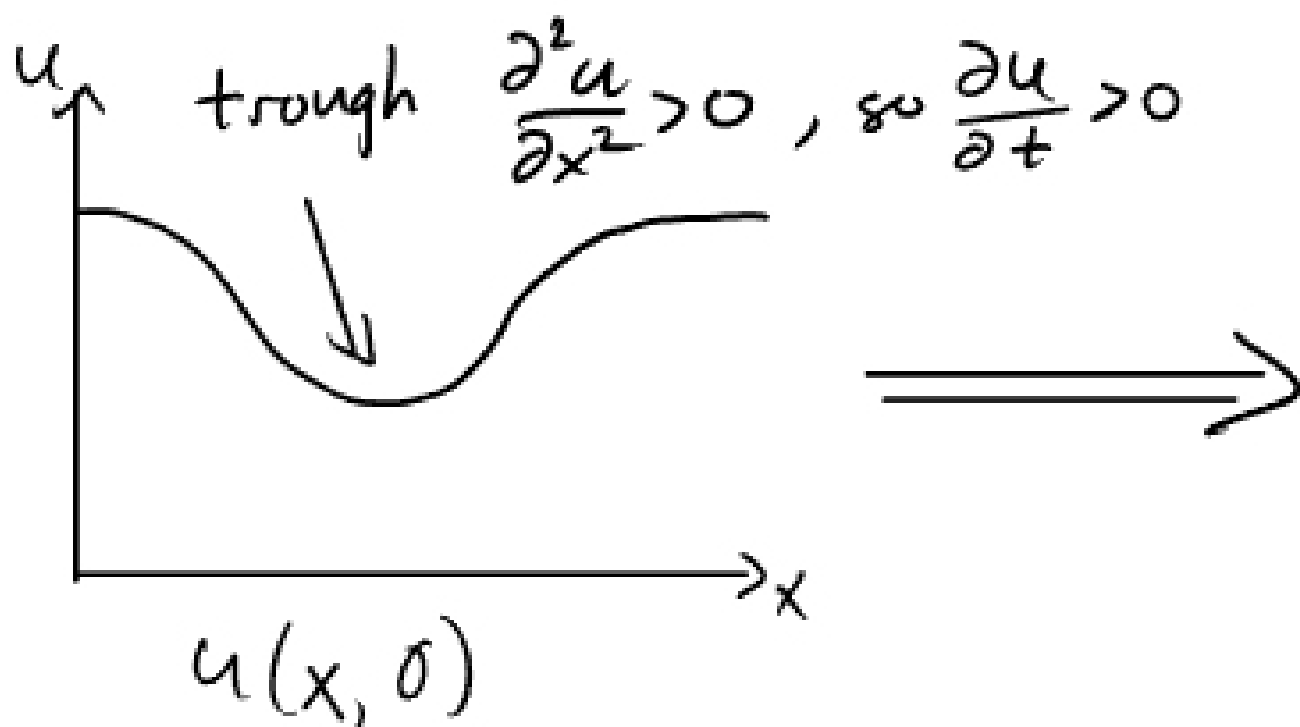
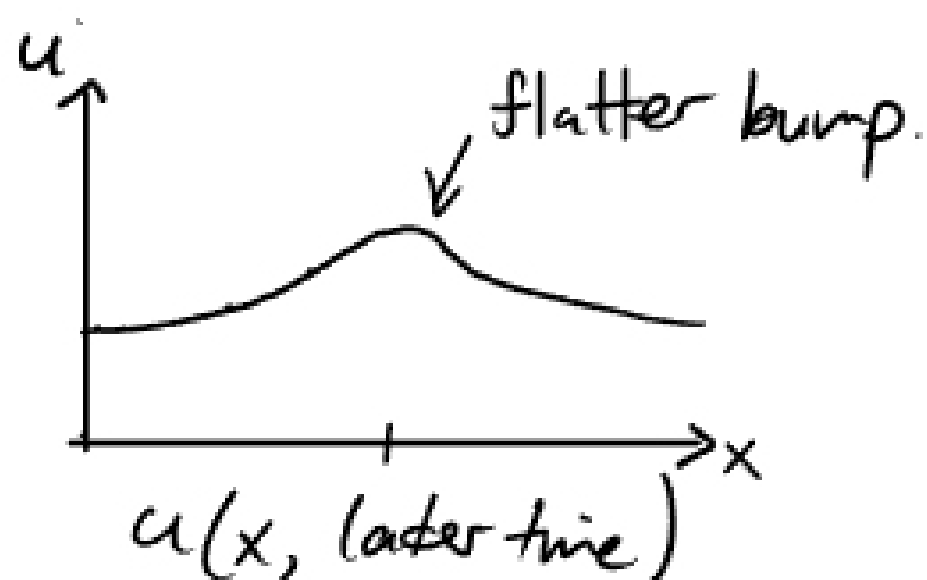
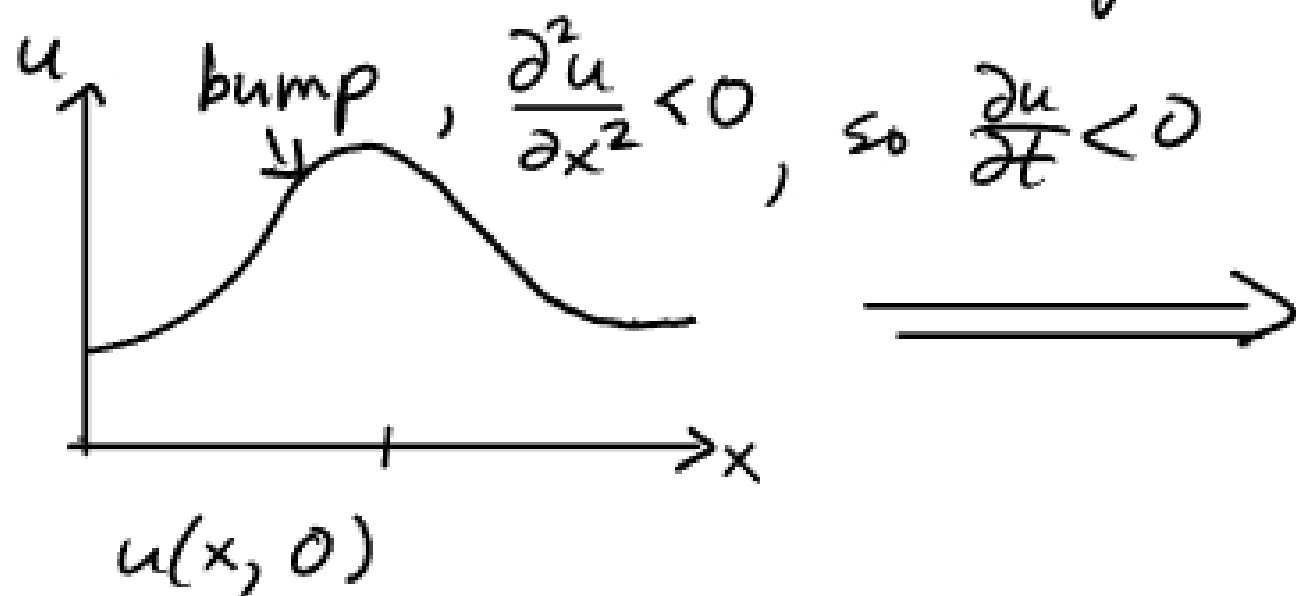
- $u(x, t)$ satisfies the heat equation if

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

- $k > 0$ is a constant called the "thermal diffusivity". It depends on the properties of the material.

Why this equation? It can be derived from the assumption that "heat flows from hot to cold."

We will not give a complete derivation, but we can understand it qualitatively



More specific setup: Boundary and initial conditions.

Let's consider a rod of length L : $0 < x < L$

$u(x,t)$ is defined for $0 < x < L$, what happens at the ends? Some possibilities:

1. Temperature is held constant: $u(0,t) = C_0$ OR

2. End is insulated, no heat escapes: $\frac{\partial u}{\partial x}(0,t) = 0$

Similar possibilities at other end $u(L,t) = C_1$ OR $\frac{\partial u}{\partial x}(L,t) = 0$

These are called "Boundary conditions". We choose them when we set up the problem.

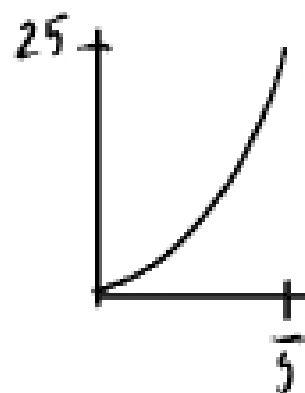
We also need to know the initial temperature when $t=0$.

This means we need to give a function $f(x)$ defined when $0 < x < L$, and we want $u(x,0) = f(x)$.

$f(x)$ is called the "initial data".

Possible "full set ups" / "Well-posed problems"

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$$\left. \begin{array}{l} \text{Domain: } 0 < x < 5 \\ \frac{\partial u}{\partial t} = 10 \frac{\partial^2 u}{\partial x^2} \quad (1) \\ u(0,t) = 0 \quad (2) \\ u(5,t) = 25 \quad (3) \\ u(x,0) = x^2 \quad (4) \end{array} \right\} \begin{array}{l} L=5 \\ (1): \text{Heat equation with } k=10 \\ (2)+(3): \text{Boundary conditions with} \\ \text{fixed temperature at endpoints} \\ (4): \text{initial temperature data.} \end{array}$$



* Domain: $0 < x < \pi$

$$\frac{\partial u}{\partial t} = 7 \frac{\partial^2 u}{\partial x^2} \quad (1)$$

$$\frac{\partial u}{\partial x}(0, t) = 0 \quad (2)$$

$$\frac{\partial u}{\partial x}(\pi, t) = 0 \quad (3)$$

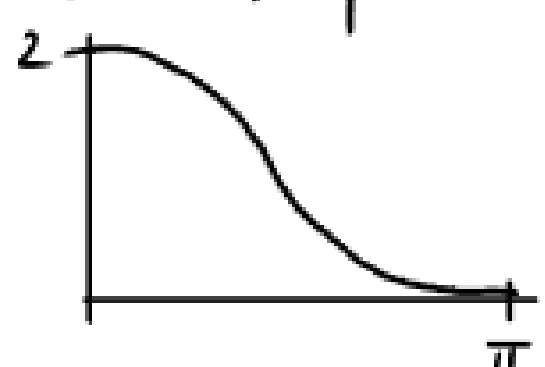
$$u(x, 0) = 1 + \cos x \quad (4)$$

$L = \pi$

(1) Heat eqn with $k = 7$

(2)+(3): Boundary conditions for insulated ends.

(4): initial temperature data



In either of these situations (and in other combinations) the differential equation determines $u(x, t)$ for all $0 < x < L$ and $t > 0$.

We want to find this solution $u(x, t)$ analytically.

Outline of the strategy

1. Find some relatively simple solutions $u(x, t) = X(x)T(t)$ "Separation of variables". This leads to an eigenvalue problem.

2. Combine simple solutions using principle of superposition. Get general solution, which is a series.

3. Use Fourier series to match initial data with the general solution.

1. Separation of variables:

Morpheus sez: "What if I told you that $u(x, t) = g(t) \sin x$ solves $\frac{\partial u}{\partial t} = 5 \frac{\partial^2 u}{\partial x^2}$?"

Then $\frac{\partial^2 u}{\partial x^2} = -g(t) \sin x$, $\frac{\partial u}{\partial t} = g'(t) \sin x$ so $g'(t) = -5g(t)$