Heat Equation II

Still trying to solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$. Last time, we sought factorizable solutions of the form $u(x,t) = X(x)T(t)$

$$\frac{dT}{dt}(t)\,X(x) = k\,\frac{d^2X}{dx^2}(x)\,T(t)$$

$$\frac{1}{kT(t)}\frac{dT}{dt}(t) = \frac{1}{X(x)}\frac{d^2X}{dx^2}(x)$$

Recall: LHS doesn't depend on x, RHS doesn't depend on t, so both are constant. We call the constant $-\lambda$.

$$\frac{1}{kT}\frac{dT}{dt} = -\lambda = \frac{1}{X}\frac{d^2X}{dx^2}$$

ie. $\frac{d^2X}{dx^2} = -\lambda X$ and $\frac{dT}{dt} = -k\lambda T$

ie. $\left\{\begin{aligned} &\frac{d^2X}{dx^2} + \lambda X = 0 \\ &\frac{dT}{dt} = -k\lambda T \end{aligned}\right\}$ for some value of λ.

These are ordinary differential equations and we know how to solve them!

$$\frac{dT}{dt} = -k\lambda T \implies T(t) = Ce^{-k\lambda t} \quad (C \text{ constant})$$

The solution of $\dfrac{d^2 X}{dx^2} + \lambda X = 0$ depend on whether $\lambda<0, \lambda=0, \lambda>0$

$\lambda > 0: \quad X(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$

$\lambda = 0: \quad X(x) = A + Bx$

$\lambda < 0: \quad X(x) = A e^{\sqrt{-\lambda}\,x} + B e^{-\sqrt{-\lambda}\,x}$

Now, $X(x)\,T(t)$ is a solution of the heat equation!

so: if $\lambda > 0$: $u = e^{-k\lambda t}\left(A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x\right)$

satisfies $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$

if $\lambda = 0$: $u = e^{-k0t}(A + Bx) = A + Bx$

satisfies it: infact, for these solutions

$\dfrac{\partial u}{\partial t} = 0$ "<u>steady-state solution</u>"

if $\lambda < 0$, let $a^2 = -\lambda$: Then

$u = e^{ka^2 t}\left(A e^{ax} + B e^{-ax}\right)$ solves it.

these solutions go to $\pm\infty$ as $t \to \infty$, so they <u>don't have much physical meaning</u>.

This is really great: we have lots of solutions to play with. There is even a free parameter λ that we can vary. We have too many! Use <u>boundary conditions</u> to cut them down

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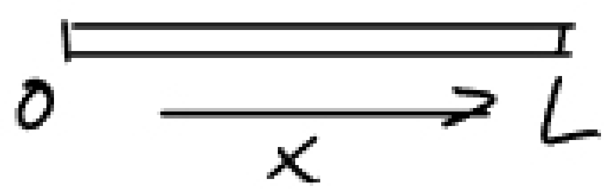
Let's consider the Rod of length L



$\ast$ Endpoints held fixed at temperature 0:

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}\right\} \text{ Boundary conditions}$$

In terms of the factorizable function $u(x,t) = X(x)T(t)$

This means $u(0,t) = X(0)T(t) = 0$

$u(L,t) = X(L)T(t) = 0$

So we need $\left.\begin{matrix} X(0)=0 \\ X(L)=0 \end{matrix}\right\}$ Endpoint conditions for $X(x)$

Thus X satisfies the endpoint value problem

$$\left\{\begin{array}{lr} \dfrac{d^2X}{dx^2} + \lambda X = 0 & (1) \\ X(0) = 0 & (2) \\ X(L) = 0 & (3) \end{array}\right\}$$ This is the eigenvalue problem we studied earlier!

Recall how to find positive eigenvalues and eigenfunctions

$$\frac{d^2X}{dx^2} + \lambda X = 0 \Longrightarrow X(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$$

$$X(0)=0 \Rightarrow A=0 \Rightarrow X(x) = B\sin\sqrt{\lambda}\,x$$

$$X(L)=0 \Rightarrow B\sin\sqrt{\lambda}\,L = 0$$

X can only be "interesting"/nontrivial if $B \neq 0$, so we must have $\sin\sqrt{\lambda}\,L = 0 \Rightarrow \sqrt{\lambda}\,L = n\pi \quad n=1,2,3,\ldots$

$$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 \quad n=1,2,3,\ldots$$

It turns out $\lambda < 0$ and $\lambda = 0$ are <u>not</u> eigenvalues.

Eigenvalues: $\lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n=1,2,3,\cdots$

Eigenfunctions: $X_n(x) = \sin\frac{n\pi x}{L}$