Heat Equation III

Today we will finish solving the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0 \\ u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

Recap: So far, we found functions $u(x,t) = X(x)T(t)$ that satisfy the heat equation and the boundary cond.
This translates into

$$\frac{dT}{dt} = -k\lambda T \qquad \left.\begin{array}{l} \dfrac{d^2X}{dx^2} + \lambda X = 0 \\ X(0) = 0 \\ X(L) = 0 \end{array}\right\} \text{eigenvalue problem}$$

Eigenvalues and Eigenfunctions:

$$\left.\begin{array}{l} \lambda_n = \left(\dfrac{n\pi}{L}\right)^2 \\ X_n(x) = \sin \dfrac{n\pi x}{L} \end{array}\right\} n = 1, 2, 3, \ldots$$

Corresponding $T(t)$'s : $T_n(t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

Basic solutions $u_n(x,t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin \dfrac{n\pi x}{L}$

$u_n(x,t)$ satisfies: $\begin{cases} \dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$ But it doesn't satisfy initial condition! $u(x,0) = f(x)$

In general, <u>none</u> of the functions $u_n(x,t)$ will satisfy initial condition. We need a more general solution!

Idea: consider

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$

<u>Proposition</u>: This $u(x,t)$ satisfies $\left\{ \begin{aligned} &\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \\ &u(0,t)=0 \\ &u(L,t)=0 \end{aligned} \right\}$

<u>Proof</u>: $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} c_n u_n\right) = \sum_{n=1}^{\infty} c_n \frac{\partial u_n}{\partial t} = \sum_{n=1}^{\infty} c_n k \frac{\partial^2 u_n}{\partial x^2}$$

b/c un is a solution

$$= k\frac{\partial^2}{\partial x^2}\left(\sum_{n=1}^{\infty} c_n u_n\right) = k\frac{\partial^2 u}{\partial x^2} \quad ✓$$

$$u(0,t) = \sum_{n=1}^{\infty} c_n u_n(0,t) = \sum_{n=1}^{\infty} c_n \cdot 0 = 0 \quad ✓$$

$$u(L,t) = \sum_{n=1}^{\infty} c_n u_n(L,t) = \sum_{n=1}^{\infty} c_n \cdot 0 = 0 \quad ✓$$

This proposition is a <u>superposition principle</u>.

So now we have a general solution which is a series

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin \frac{n\pi x}{L}$$

As with any general solution, we can determine the constants c_n from the initial data:

We want to match $u(x,0)$ with some given function $f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 \cdot 0} \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$$

Observe: this is a Fourier sine series of period $2L$!

We want it to equal $f(x)$ for $0<x<L$

WANT TO MATCH: $f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$, $0<x<L$

This is precisely the meaning of "the Fourier sine series of $f(x)$". So we know how to find the coefficients c_n:

Conceptually: extend $f(x)$ to an odd periodic function of period $2L$; take Fourier series

Formulaically: $c_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx$

If we want to, we can write a big honking formula for the solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right] e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\frac{n\pi x}{L}$$

solves (1) $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, (2) $u(0,t)=0$, (3) $u(L,t)=0$, <u>and</u>

(4) $u(x,t) = f(x)$