Heat equation variations ; Start Wave Equation.

Last time we finished solving the Heat equation in the case where the ends of the rod have temperature held constant at zero. There are several other versions of the problem that can be solved by a similar method.

Variation I: Insulated Ends. We will only sketch this, you should fill in the details.

$$\left\{\begin{array}{ll} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} & \\ \dfrac{\partial u}{\partial x}(0,t)=0 & \multirow{2}{*}{} \\ \dfrac{\partial u}{\partial x}(L,t)=0 & \left.\right\} \text{insulated ends} \\ u(x,0)=f(x) & \text{initial data} \end{array}\right.$$

Separation of variables $u(x,t)=X(x)T(t)$ translates problem into

$$\frac{dT}{dt} = -k\lambda T \qquad \frac{d^2X}{dx^2}+\lambda X=0$$

$$\frac{dX}{dx}(0)=0$$

$$\frac{dX}{dx}(L)=0$$

The eigenvalue problem for X has the solution

Eigenvalues $\lambda_0 = 0$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n=1,2,3,\ldots$

Eigenfunctions $X_0 = 1$, $X_n = \cos\frac{n\pi x}{L}$, $n=1,2,3,\ldots$

Or, since $\cos 0 = 1$, we may write this more succinctly as

$$\left.\begin{array}{l} \lambda_n = \left(\dfrac{n\pi}{L}\right)^2 \\ X_n = \cos\dfrac{n\pi x}{L} \end{array}\right\} \quad n=\underline{0},1,2,3,\ldots$$

$n=0$ is included.

The corresponding T-factors are $T_n = e^{-k\left(\frac{n\pi}{L}\right)^2 t}$, $n = \underline{0}, 1, 2, 3, \ldots$

Basic solutions $u_n(x,t) = T_n(t)\,X_n(x) = e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos \frac{n\pi x}{L}$

$n = \underline{0}, 1, 2, 3, \ldots$

Observe that $u_0(x,t) = 1$ is a constant function.

The boundary conditions $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 0$
are "linear homogeneous"
The principle of superposition holds.

General solution: $u(x,t) = \sum_{n=\underline{0}}^{\infty} c_n u_n(x,t) = \underline{c_0} + \sum_{n=1}^{\infty} c_n u_n(x,t)$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos \frac{n\pi x}{L}$$

To fit initial data: $u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi x}{L} \overset{???}{=} f(x)$

This looks like Fourier cosine series of period $2L$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \qquad (0 < x < L)$$

where $a_0 = \frac{2}{L}\int_0^L f(x)\,dx \qquad a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\,dx$

In order to match initial data, put $c_0 = \frac{a_0}{2}$, $c_n = a_n$ for $n = 1, 2, 3, \ldots$

<u>Example</u>

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} \\ &\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(1,t) = 0 \\ &u(x,0) = 1 + \cos \pi x \end{aligned}\right\}$$

<u>soln</u>:

$$\boxed{u(x,t) = 1 + e^{-3\pi^2 t} \cos \pi x}$$

Variation II: Nonhomogeneous boundary conditions:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

$u(0,t) = A$

$u(L,t) = B$

$u(x,0) = f(x)$

The boundary conditions $u(0,t)=A$, $u(L,t)=B$ Ruin the principle of superposition, it doesn't hold.

General idea: find a particular solution u_0 to the heat eqn and boundary conditions.

Consider difference $u-u_0$. It satisfies <u>homogeneous</u> boundary conditions, and we can use the previous techniques. <u>See problem 3</u> on homework 6.

Wave equation: For waves in water, air, vibrating string, ...

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

$u(x,t) =$ water level, air pressure, displacement of vibrating medium.

We will solve the vibrating string using separation of variables.

But first:

To break up the monotony a bit, let's look at the wave eqn without boundary conditions, i.e., x ranges over the whole real line.

There is an amazingly elegant solution found by D'Alembert:

Let $F(z)$ and $G(z)$ be functions (assume F'' and G'' exist)

Then

$$u(x,t) = F(x-ct) + G(x+ct)$$

Satisfies $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$