Laplace equation I

We consider functions of two variables $u(x,y)$, where now both x and y are thought of as <u>spatial</u> variables.

The <u>two-dimensional Laplace equation</u> is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y^2} = 0$$

Solutions of Laplace equation are also called <u>harmonic functions</u>.

Similarly, the <u>three-dimensional Laplace equation</u> is for a function $u(x,y,z)$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Note that

$$\operatorname{div}(\operatorname{grad} u) = \nabla \cdot (\nabla u) = \nabla \cdot \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\rangle$$
$$= \frac{\partial u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Also written $\nabla^2 u$ or Δu, called the <u>Laplacian</u> of u.
The operator $\Delta = \nabla^2 = \operatorname{div} \operatorname{grad} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the <u>Laplacian</u> or <u>Laplace operator</u>.

In 2d: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. In 1d: $\Delta = \frac{d^2}{dx^2}$

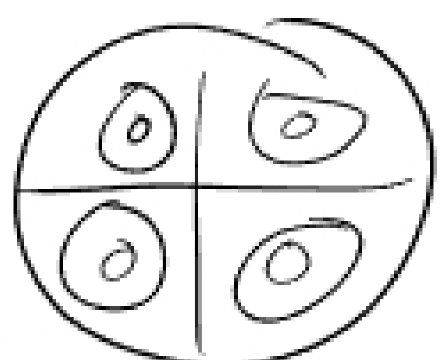
This is one of the most important operators in mathematics.

The heat equation in 2 or 3 spatial dimensions and 1 time dimension is $\frac{\partial u}{\partial t} = k \Delta u$

Think of heat in plate  or ball 

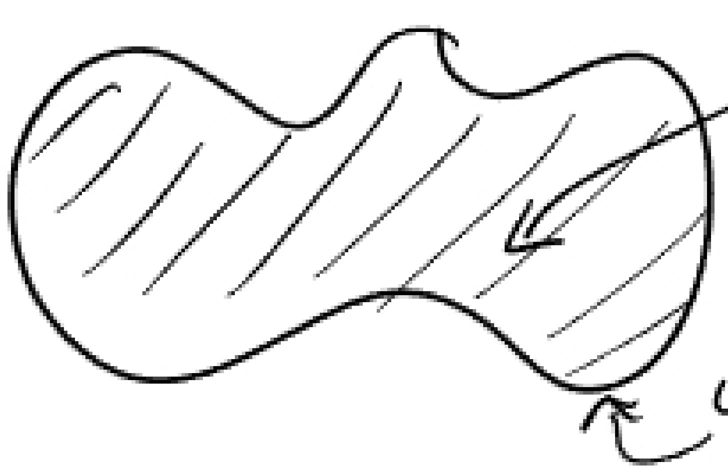
The wave equation in 2 or 3 spatial dimensions and 1 time dim.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

Think vibrations in a drum head 

Steady state temperature: $\frac{\partial u}{\partial t} = k \Delta u$ and $\frac{\partial u}{\partial t} = 0$,

Hence $\Delta u = 0$. So Laplace's equation describes temperature distribution in equilibrium. But we still need a boundary condition: We can specify u on the boundary of the domain

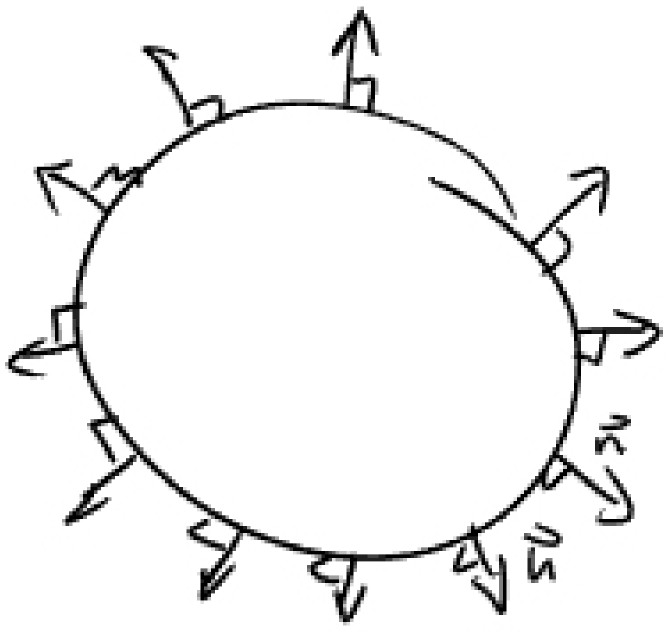


$\Delta u = 0$ inside domain

$u(x,y) =$ given $g(x,y)$ on boundary

These are called <u>Dirichlet</u> boundary conditions (keep temperature on the boundary fixed.)

There are also Neumann boundary conditions, where we require $\vec{n} \cdot \nabla u = 0$ along boundary, where $\vec{n}$ is the normal vector to the boundary.



Require directional derivative $\vec{n} \cdot \nabla u = 0$
In heat case this corresponds to insulated boundary.

Other applications: Electrostatic potential $V(x,y,z)$
Electric field $\vec{E} = \nabla V$.
Gauss' law $\nabla \cdot \vec{E} = \rho$ where ρ is charge density.
If $\rho = 0$:

$$0 = \nabla \cdot \vec{E} = \nabla \cdot (\nabla V) = \nabla^2 V = \Delta V$$

So V satisfies Laplace's equation in regions where no charge is present.

Example situation:

V is specified on boundary of sphere

$\Delta V = 0$ inside.

In this course we will solve Laplace's equation on a rectangle with Dirichlet boundary conditions.