MATH 433

Applied Algebra

Lecture 13:
Examples of groups.

Abstract groups

Definition. A **group** is a set G, together with a binary operation $*$, that satisfies the following axioms:

(G1: closure)
for all elements g and h of G, $g * h$ is an element of G;

(G2: associativity)
$(g * h) * k = g * (h * k)$ for all $g, h, k \in G$;

(G3: existence of identity)
there exists an element $e \in G$, called the **identity** (or **unit**) of G, such that $e * g = g * e = g$ for all $g \in G$;

(G4: existence of inverse)
for every $g \in G$ there exists an element $h \in G$, called the **inverse** of g, such that $g * h = h * g = e$.

The group $(G, *)$ is said to be **commutative** (or **Abelian**) if it satisfies an additional axiom:

(G5: commutativity) $g * h = h * g$ for all $g, h \in G$.

Basic examples. • Real numbers $\mathbb{R}$ with addition.

(G1) $x, y \in \mathbb{R} \implies x + y \in \mathbb{R}$

(G2) $(x + y) + z = x + (y + z)$

(G3) the identity element is 0 as $x + 0 = 0 + x = x$

(G4) the inverse of x is $-x$ as $x + (-x) = (-x) + x = 0$

(G5) $x + y = y + x$

• Nonzero real numbers $\mathbb{R} \setminus \{0\}$ with multiplication.

(G1) $x \neq 0$ and $y \neq 0 \implies xy \neq 0$

(G2) $(xy)z = x(yz)$

(G3) the identity element is 1 as $x1 = 1x = x$

(G4) the inverse of x is x^{-1} as $xx^{-1} = x^{-1}x = 1$

(G5) $xy = yx$