MATH 433

Applied Algebra

Lecture 16:
Algebraic structures (continued).

Ring

Definition. A **ring** is a set R, together with two binary operations usually called **addition** and **multiplication** and denoted accordingly, such that

- R is an Abelian group under addition,
- R is a semigroup under multiplication,
- multiplication distributes over addition.

A ring R is called **commutative** if the multiplication is commutative. R is called a **ring with identity** if there exists an identity element for multiplication (denoted 1).

An **integral domain** is a nontrivial commutative ring with identity and no zero-divisors (i.e., $ab = 0$ implies $a = 0$ or $b = 0$).

Examples of rings

- Real numbers $\mathbb{R}$.
- Integers $\mathbb{Z}$.
- $2\mathbb{Z}$: even integers.
- $\mathbb{Z}_n$: congruence classes modulo n.
- $\mathcal{M}_n(\mathbb{R})$: all $n \times n$ matrices with real entries.
- $\mathcal{M}_n(\mathbb{Z})$: all $n \times n$ matrices with integer entries.
- $\mathcal{M}_n(R)$: all $n \times n$ matrices with entries from a ring R.
- $\mathbb{R}[X]$: polynomials in variable X with real coefficients.
- $\mathbb{Z}[X]$: polynomials in variable X with integer coefficients.
- $R[X]$: polynomials in variable X with coefficients from a ring R.
- $\mathbb{R}(X)$: rational functions in variable X with real coefficients.
- All functions $f : \mathbb{R} \to \mathbb{R}$.