AAE 340

Lecture #19

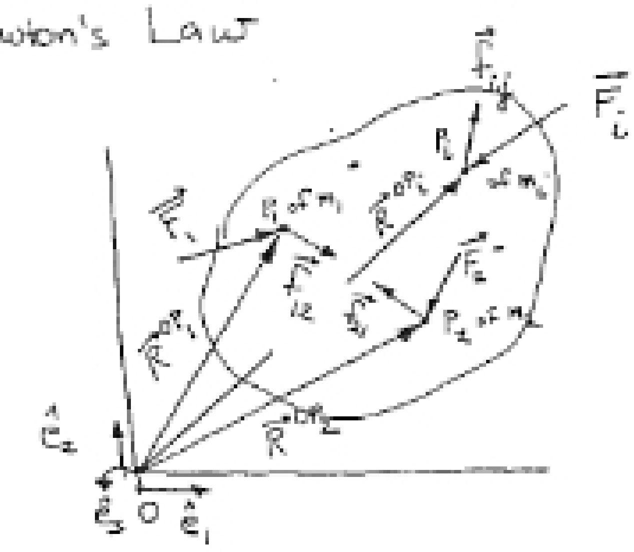
CHAPTER 6

SYSTEMS OF PARTICLES

Euler extended Newton's Laws
from single particle to the rigid body.
Must average effect of many particles

Equations of Motion for a System of Particles

1. Newton's Law



$$\vec{F} = m\,{}^{e}\vec{a}^{OP} = m\,{}^{e}\ddot{\vec{R}}^{OP} \qquad (1)$$

Law for one particle, P.

For system:

$\vec{F}_i$ = External force on i^{th} Particle

$\vec{f}_{ij}$ = Internal force on i^{th} Particle from j^{th} Particle

where we assume that $\vec{f}_{ij}$ and $\vec{f}_{ji}$ are COLLINEAR.

Newton's Law (1) for i^{th} particle:

$$\vec{F}_i + \sum_{j=1}^{n} \vec{f}_{ij} = m_i\,{}^{e}\ddot{\vec{R}}^{OP_i} \qquad (2)$$

Summing the forces on all the particles:

$$\sum_{i=1}^{n} \vec{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \vec{f}_{ij} = \sum_{i=1}^{n} m_i\,{}^{e}\ddot{\vec{R}}^{OP_i} \qquad (3)$$

Using Newton's 3RD Law (Action - Reaction)

$$\vec{f}_{ij} = -\vec{f}_{ji} \qquad (4)$$

The total internal force a particle can exert on itself is zero

$$\vec{f}_{ii} = 0 \qquad (5)$$

But

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \vec{f}_{ij} = 0 \qquad (6)$$

The total ext. force on the system is

$$\vec{F} = \sum_{i=1}^{n} \vec{F}_i \qquad (7)$$

The total mass is

$$m = \sum_{i=1}^{n} m_i \qquad (8)$$

The center of mass is (rel. to O)

$$\vec{R}^{OC} = \frac{1}{m} \sum_{i=1}^{n} m_i \vec{R}^{OP_i} \qquad (9)$$

Velocity of c.m.

$${}^{e}\dot{\vec{R}}^{OC} = \frac{1}{m} \sum_{i=1}^{n} m_i\,{}^{e}\dot{\vec{R}}^{OP_i} \qquad (10)$$

Acc. of c.m.

$${}^{e}\ddot{\vec{R}}^{OC} = \frac{1}{m} \sum_{i=1}^{n} m_i\,{}^{e}\ddot{\vec{R}}^{OP_i} \qquad (11)$$

From 3, 6, 7 and 11

$$\boxed{\vec{F} = m\,{}^{e}\ddot{\vec{R}}^{OC}} \qquad (12)$$

Newton's Law for Sys. of Particles

Eq^n 12 states that the sum of the ext. forces acting on sys. accelerates the center of mass as if all mass were concentrated @ one pt., namely the cm.

2. Euler's Law (Fixed Ref. pt.)

$$\vec{M}^{O} = {}^{e}\dot{\vec{H}}^{O} \qquad (13)$$

for particle.

For system, find Euler's Law for <u>fixed ref. pt, O</u>.

Total angular momentum

$${}^{e}\vec{H}^{O} = \sum_{i=1}^{n} \vec{R}^{OP_i} \times m_i\, {}^{e}\dot{\vec{R}}^{OP_i} \qquad (14)$$

Differentiating (wrt e)

$${}^{e}\dot{\vec{H}}^{O} = \sum_{i=1}^{n} \vec{R}^{OP_i} \times \underbrace{m_i\, {}^{e}\ddot{\vec{R}}^{OP_i}}_{\text{sub. Eq. 2}} + 0 \qquad (15)$$

$${}^{e}\dot{\vec{H}}^{O} = \underbrace{\sum_{i=1}^{n}\Big[\vec{R}^{OP_i} \times \vec{F}_i}_{\vec{M}^{O}} + \vec{R}^{OP_i} \times \sum_{j=1}^{n} \vec{f}_{ij}\Big] \qquad (16)$$

Note:

$$\vec{M}^{O} = \sum_{i=1}^{n} \vec{R}^{OP_i} \times \vec{F}_i \qquad (17)$$

total moment on system due to external forces.

Show 2^{nd} term in (16) on RHS is zero

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \vec{R}^{OP_i} \times \vec{f}_{ij} = 0 \qquad (18)$$

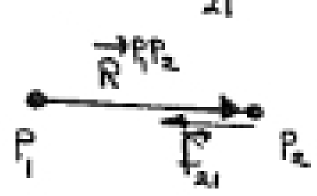
because $\vec{f}_{ij} = -\vec{f}_{ji}$ and because they are <u>collinear</u>!

Let's consider the simplest case in which we have just 2 particles:

$$\sum_{i=1}^{2}\sum_{j=1}^{2} \vec{R}^{OP_i} \times \vec{f}_{ij} = \vec{R}^{OP_1} \times \cancel{\vec{f}_{11}}^{\,0} + \vec{R}^{OP_1} \times \vec{f}_{12} + \vec{R}^{OP_2} \times \vec{f}_{21} + \vec{R}^{OP_2} \times \cancel{\vec{f}_{22}}^{\,0}$$

Next we can write

$$\begin{aligned}
&\vec{R}^{OP_1} \times \vec{f}_{12} + \vec{R}^{OP_2} \times \vec{f}_{21} \\
&= \vec{R}^{OP_1} \times \vec{f}_{12} + \underbrace{(\vec{R}^{OP_1} + \vec{R}^{P_1P_2}) \times \vec{f}_{21}}_{-\vec{R}^{OP_1} \times \vec{f}_{12} + \vec{R}^{P_1P_2} \times \vec{f}_{21}} \\
&\qquad \underbrace{\qquad\qquad\qquad}_{=0} \\
&= \vec{R}^{P_1P_2} \times \vec{f}_{21} = 0
\end{aligned}$$



A similar analysis holds for an arbitrary number of particles.

Thus, from Eqs. (16)-(18) we have

$$\boxed{{}^{e}\dot{\vec{H}}^{O} = \vec{M}^{O} \quad (19) \qquad \text{Euler's Law for Fixed Ref. Point, O}}$$

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Lecture #20

SYSTEMS OF PARTICLES (cont.)

Last time we derived Euler's Law for a Fixed Reference Point, O:

$$\boxed{\vec{M}^{O} = {}^{e}\dot{\vec{H}}^{O} \qquad (19)}$$

Euler's law can be extended to include:

a) c.m. as reference point

b) arbitrary reference point

To find law for c.m. as ref pt., write (17) as

$$\vec{M}^{O} = \sum_{i=1}^{n} \vec{R}^{OC} \times \vec{F}_i + \underbrace{\vec{R}^{CP_i} \times \vec{F}_i}_{\vec{M}^{C}}$$

$$= \vec{R}^{OC} \times \vec{F} + \vec{M}^{C} \qquad (20)$$

Write (15) as

$${}^{e}\dot{\vec{H}}^{O} = \underbrace{\sum_{i=1}^{n} \vec{R}^{OC} \times m_i \, {}^{e}\ddot{\vec{R}}^{OP_i}}_{(2),\,F(n)} + \vec{R}^{CP_i} \times m_i \, {}^{e}\ddot{\vec{R}}^{OP_i}$$

$$= \vec{R}^{OC} \times \vec{F} + \sum_{i=1}^{n} \vec{R}^{CP_i} \times m_i \left({}^{e}\ddot{\vec{R}}^{OC} + {}^{e}\ddot{\vec{R}}^{CP_i} \right) \qquad (21)$$

Equating 20 = 21

$$\vec{M}^{C} = \underbrace{\sum_{i=1}^{n} \vec{R}^{CP_i} \times m_i \, {}^{e}\ddot{\vec{R}}^{OC}}_{=0} + \underbrace{\sum_{i=1}^{n} \vec{R}^{CP_i} \times m_i \, {}^{e}\ddot{\vec{R}}^{CP_i}}_{{}^{e}\dot{\vec{H}}^{C}}$$

$\left(\text{because } \sum_{i=1}^{n} m_i \vec{R}^{CP_i} = 0\right)$ (by analogy, (15))

Thus we have

$$\boxed{\vec{M}^{C} = {}^{e}\dot{\vec{H}}^{C} \qquad (22)}$$

Euler's Law for c.m. ref. pt.

For Arbitrary Ref. Pt, q

$$\boxed{{}^{e}\dot{\vec{H}}^{q} = \vec{M}^{q} + \underbrace{\vec{R}^{qc}}_{\text{moment arm from q to c.m.}} \times \underbrace{\left(-m \, {}^{e}\ddot{\vec{R}}^{Oq}\right)}_{\text{can be thought of as inertial force due to acc. of q. } (-m\,{}^{e}\ddot{\vec{R}}^{Oq})} \qquad (23)}$$

INTEGRALS OF THE EOM's FOR SYSTEMS OF PARTICLES

Principle of Work and Energy

From Eq. 28, p. 96, Work is

$$W = \int_{t_1}^{t_2} \vec{F} \cdot {}^{e}\vec{V}^{OP} \, dt \Big|_{e} \qquad (24)$$

where e = inertial frame