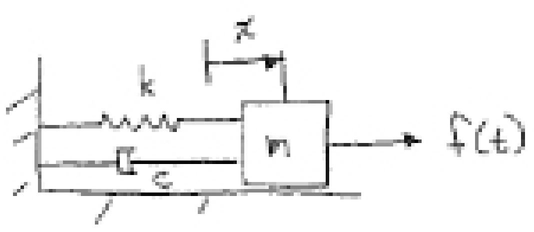
AAE 340
Lecture # 7

Exam 1

Forced Vibration of Mass-Spring-Damper System (Case 3)



Case 1: Unforced, Undamped $c = 0$, $f(t) = 0$

Case 2: Unforced with Damping: $f(t) = 0$
- underdamped $0 < \zeta < 1$
- critically damped $\zeta = 1$
- overdamped $\zeta > 1$

Case 3: Forced with Damping

Assume
$$f(t) = f_0 \cos \omega t \qquad (88)$$

EOM from (1) and (88)

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{f_0}{m}\cos \omega t \qquad (89)$$

Or in Classical Form (39):

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f_0}{m}\cos \omega t \qquad (90)}$$

Sol^n as sum of two parts:

$$x = x_t + x_s \qquad (91)$$

where

x_t = transient sol^n, complementary f^n or homogeneous sol^n (x_h)

of the homogeneous eq^n (39)

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

Which we have solved for all cases of ζ.

x_s = steady state sol^n (or particular integral, x_p) which satisfies the complete d.e. (90).

The general transient sol^n has the form:

$$x_t = C_1 e^{s_1 t} + C_2 e^{s_2 t} \qquad (92)$$

when $s_1 \neq s_2$ and the roots are given by (42);

or

$$x_t = C_1 e^{-\omega_n t} + t\, C_2 e^{-\omega_n t} \qquad (61)$$

when $s_1 = s_2$ ($\zeta = 1$).

Steady State Sol^n, x_s

When $f(t) = c$, t^n (n = pos. integer), e^{kt}, $\sin \omega t$, $\cos \omega t$

we use method of undetermined coefficients:

$$\text{Put} \quad x_s(t) = C_0 f(t) + C_1 f'(t) + C_2 f''(t) + \cdots \qquad (93)$$

Thus:

$$x_s(t) = C_0 \cos \omega t + C_1 \sin \omega t \qquad (94)$$

or, equivalently:

$$x_s(t) = A\cos(\omega t + \phi) \qquad (95)$$

$$\Rightarrow \dot{x}_s(t) = -A\omega \sin(\omega t + \phi) \qquad (96)$$

$$\ddot{x}_s(t) = -A\omega^2 \cos(\omega t + \phi) \qquad (97)$$

Subs. 95-97 into 90:

$$-A\omega^2\overset{①}{\cos}(\omega t+\phi) - 2\zeta\omega_n\omega A\sin(\omega t+\phi) + \omega_n^2 A\overset{①}{\cos}(\omega t+\phi) = \frac{f_0}{m}\cos\omega t \tag{98}$$

Using the trig. id. (12) on page 16

$$\cos(\omega t+\phi) = \cos\omega t\cos\phi - \sin\omega t\sin\phi \tag{99}$$

Similarly

$$\sin(\omega t+\phi) = \sin\omega t\cos\phi + \cos\omega t\sin\phi \tag{100}$$

Eq^n (98) becomes (after sub. 99 + 100):

$$(\omega_n^2-\omega^2)A\left[\cos\omega t\cos\phi - \sin\omega t\sin\phi\right] - 2\zeta\omega_n\omega A\left[\sin\omega t\cos\phi + \cos\omega t\sin\phi\right] = \frac{f_0}{m}\cos\omega t \tag{101}$$

Collecting terms in $\cos\omega t$:

$$A\cos\omega t\left[(\cos\phi)(\omega_n^2-\omega^2) - 2\zeta\omega_n\omega\sin\phi\right] = \frac{f_0}{m}\cos\omega t \tag{102}$$

and collecting terms in $\sin\omega t$

$$A\sin\omega t\left[-(\omega_n^2-\omega^2)\sin\phi - 2\zeta\omega_n\omega\cos\phi\right] = 0 \tag{103}$$

From (103)

$$\tan\phi = \frac{2\zeta\omega_n\omega}{\omega^2-\omega_n^2}$$

or

$$\boxed{\phi = \tan^{-1}\left[\frac{-2\zeta(\omega/\omega_n)}{1-(\omega^2/\omega_n^2)}\right]} \tag{104}$$

<u>Phase angle</u>

From (102):

$$A = \frac{f_0/m}{(\omega_n^2-\omega^2)\cos\phi - 2\zeta\omega_n\omega\sin\phi}$$

$$= \frac{f_0/m\cos\phi}{(\omega_n^2-\omega^2) - 2\zeta\omega_n\omega\tan\phi}$$

$$= \frac{\sec\phi\, f_0/m}{(\omega_n^2-\omega^2) - 2\zeta\omega_n\omega\left(\frac{2\zeta\omega_n\omega}{\omega^2-\omega_n^2}\right)} \tag{105}$$

From the trig id

$$\sec\phi = \sqrt{\tan^2\phi+1} = \sqrt{\left(\frac{2\zeta\omega_n\omega}{\omega^2-\omega_n^2}\right)^2+1} \tag{106}$$

we obtain

$$A = \frac{\sqrt{\frac{(2\zeta\omega_n\omega)^2+(\omega^2-\omega_n^2)^2}{\cancel{(\omega^2-\omega_n^2)^2}\ ①}}\ f_0/m}{\frac{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}{\cancel{\omega_n^2-\omega^2}\ ①}}$$

$$= \frac{(f_0/m)\,\mathrm{sgn}(\omega_n-\omega)}{\sqrt{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}} \tag{107}$$

Note: $\dfrac{\omega_n^2-\omega^2}{\sqrt{(\omega^2-\omega_n^2)^2}} = \mathrm{sgn}(\omega_n-\omega)$

Thus,

$$x_s(t) = \frac{(f_0/m\omega_n^2)\,\mathrm{sgn}(\omega_n-\omega)\cos(\omega t+\phi)}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2+\left[2\zeta(\omega/\omega_n)\right]^2}} \tag{108}$$

where ϕ is given by (104)

$$\phi = \tan^{-1}\left[\frac{-2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right] \tag{104}$$

$\equiv$ <u>phase angle</u>

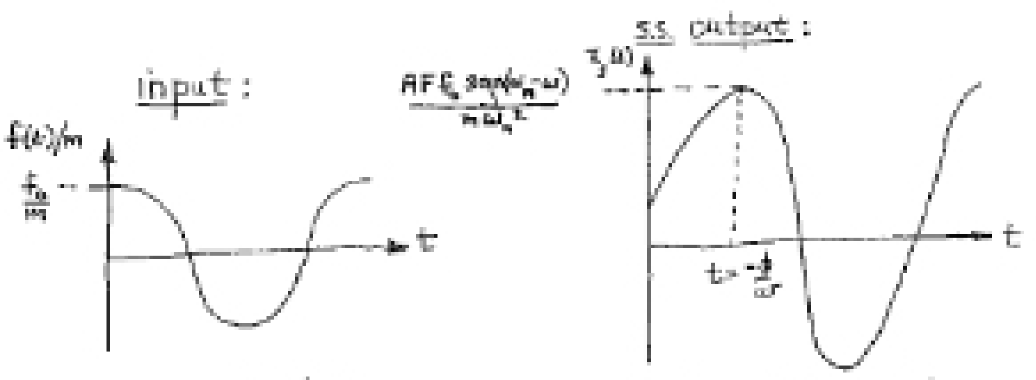
The <u>amplification factor</u>, AF, is defined as

$$\boxed{AF \equiv \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}}} \tag{109}$$

<u>Notes</u> $\mathrm{sgn}(z) = 1$ if $z \geq 0$
$= -1$ if $z < 0$

In the special case of $\omega = \omega_n$, $A = f_0/(2m\zeta\omega_n^2)$ and $\phi = -\pi/2$ so we need $\mathrm{sgn}(0) \equiv 1$. (Of course $\zeta \neq 0$.)

The Meaning of Amplification Factor (AF) and Phase Angle (ϕ)



sinusoidal:

$\frac{f_o}{m}\cos\omega t$

↑

constant amplitude

steady state output is sinusoidal:
same frequency (period),
different magnitude,
different phase:

$$\frac{f_o}{m\omega_n^2} AF\, \mathrm{sgn}(\omega_n-\omega)\cos(\omega t+\phi)$$

Note: to find the complete sol^n, add $x_s + x_t$ (steady state plus transient sol^{ns}) and then evaluate C_1 and C_2 from I.C.'s.

If we set $\omega = 0 \rightarrow$ static deflection problem and we find

$$AF = 1, \quad \phi = 0$$

and

$$x_s = \frac{f_o}{m\omega_n^2} = \frac{f_o}{k} = \text{static deflection}$$

Thus, since in general

$$x_s = \frac{f_o}{k} AF \cos(\omega t + \phi)\,\mathrm{sgn}(\omega_n - \omega)$$

we see that the AF can be viewed as the ratio of the steady-state amplitude to the static-deflection amplitude

$$AF = \frac{|x_s|}{\frac{f_o}{k}}$$