AAE 340
Lecture #13

Ch. 5 cont.

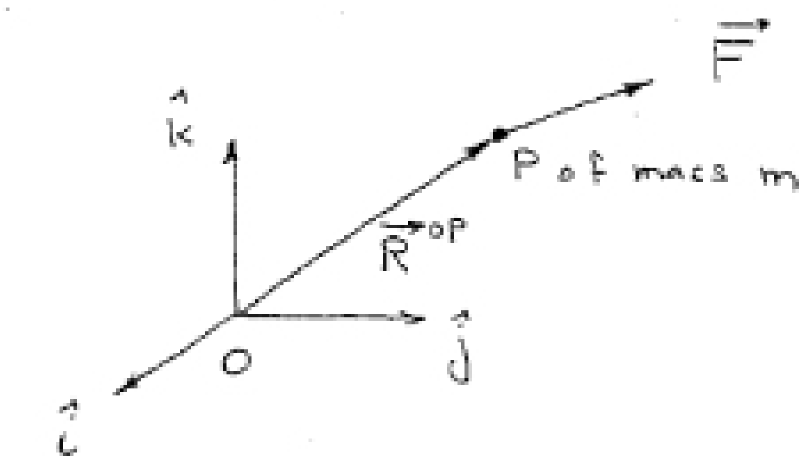
Derivation of Euler's Law from Newton's Law

$$\boxed{\vec{M}^{O} = {}^{i}\frac{d\vec{H}^{O}}{dt} \qquad (20)}$$

Euler's Law

$\vec{M}^{O}$ = moment about O
$\vec{H}^{O}$ = angular momentum about O
where O is a fixed point in an inertial frame.

Derivation:



$$\vec{F} = m\,{}^{i}\frac{d^2\vec{R}^{OP}}{dt^2} \qquad (21)$$

Take cross product

$$\vec{R}^{OP} \times \vec{F} = \vec{R}^{OP} \times m\,{}^{i}\frac{d^2\vec{R}^{OP}}{dt^2} \qquad (22)$$

and note that

$$\vec{R}^{OP} \times {}^{i}\frac{d^2\vec{R}^{OP}}{dt^2} = {}^{i}\frac{d}{dt}\left[\vec{R}^{OP} \times {}^{i}\frac{d\vec{R}^{OP}}{dt}\right] \qquad (23)$$

because

$${}^{i}\frac{d}{dt}\left[\vec{R}^{OP} \times {}^{i}\frac{d\vec{R}^{OP}}{dt}\right]$$

$$= \cancelto{0}{{}^{i}\frac{d}{dt}\vec{R}^{OP} \times {}^{i}\frac{d}{dt}\vec{R}^{OP}} + \vec{R}^{OP} \times {}^{i}\frac{d^2\vec{R}^{OP}}{dt^2}$$

Since

$$\vec{M}^{O} \equiv \vec{R}^{OP} \times \vec{F} \qquad (24)$$

moment of force

and

$$\vec{H}^{O} \equiv \vec{R}^{OP} \times m\,{}^{i}\frac{d\vec{R}^{OP}}{dt} \qquad (25)$$

moment of momentum or angular momentum

Eq^n (22) - (25) give

$$\boxed{\vec{M}^{O} = {}^{i}\dot{\vec{H}}^{O}} \quad \checkmark$$

Integrals of the Motion

Goals: find EOM, solve for R(t). Integrals ease integration problem.

Time Integrals:

$$\underbrace{\vec{\mathcal{F}}}_{\text{linear impulse}} = \int_{t_1}^{t_2} \vec{F}\,dt\,\Big|_i = \int_{t_1}^{t_2} m\,{}^{i}\frac{d}{\cancel{dt}}\left({}^{i}\frac{d}{dt}\vec{R}^{OP}\right)\cancel{dt}\,\Big|_i$$

$$= \underbrace{m\,{}^{i}\frac{d\vec{R}^{OP}(t_2)}{dt} - m\,{}^{i}\frac{d\vec{R}^{OP}(t_1)}{dt}}_{\text{change in linear momentum}} \qquad (26)$$

Eq^n (26) states the Principle of Linear Impulse and Linear Momentum.

When $\vec{\mathcal{F}} = 0$ (or any comp. = 0) we have conservation of Lin. Mom.
⇒ a useful constant.

Time integral of (20) provides

$$\underbrace{\vec{\mathcal{M}}^{o}}_{\text{angular impulse}} = \int_{t_1}^{t_2} \vec{M}^{o}\,dt = \underbrace{\vec{H}^{o}(t_2) - \vec{H}^{o}(t_1)}_{\text{Change of Ang. Mom.}} \qquad (27)$$

Principle of Angular Impulse and Angular Momentum

When any comp. of $\vec{\mathcal{M}}^{o}$ is zero then we have conservation of angular momentum in that component.

Another time integral is

$$\int_{t_1}^{t_2} \vec{F}\cdot{}^{i}\vec{V}^{op}\,dt\,\Big|_{i}$$

$$= \int_{t_1}^{t_2} \vec{F}\cdot\frac{{}^{i}d\vec{R}^{op}}{dt}\,dt$$

$$= \int_{R_1}^{R_2} \vec{F}\cdot{}^{i}d\vec{R}^{op}$$

$$= W \;\begin{array}{l} = \text{work} \\ = \text{scalar} \end{array} \qquad (28)$$

where ${}^{i}d\vec{R}^{op}$ is a differential element along a path in inertial space traveled by the particle.

Note that

$$\underbrace{\vec{F}\cdot{}^{i}\vec{V}^{op}}_{\text{scalar}} = m\,\frac{{}^{i}d\,{}^{i}\vec{V}^{op}}{dt}\cdot{}^{i}\vec{V}^{op}$$

$$= \frac{1}{2}m\,\frac{d}{dt}\Big[\underbrace{{}^{i}\vec{V}^{op}\cdot{}^{i}\vec{V}^{op}}_{\text{a scalar}}\Big] \qquad (29)$$

From Eq^{ns} 28 & 29 we have

The Principle of Work and Kinetic Energy

$$W = \int_{t_1}^{t_2} \vec{F}\cdot{}^{i}\vec{V}^{op}\,dt$$

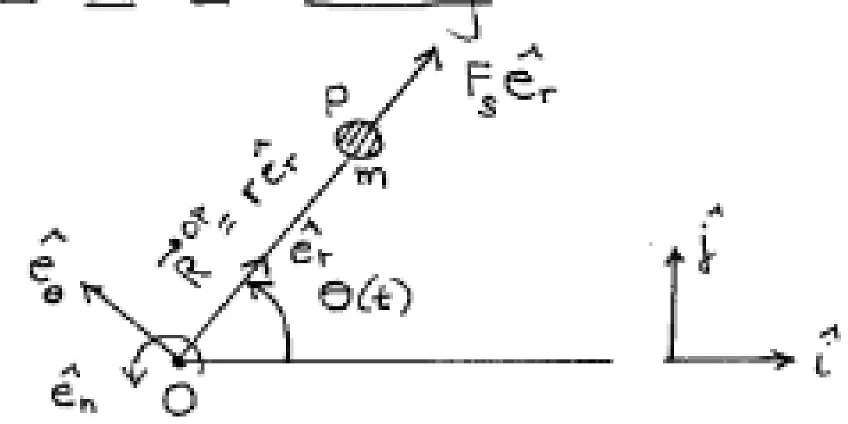
$$= \frac{1}{2}m\int_{t_1}^{t_2} \frac{d}{dt}\left[{}^{i}\vec{V}^{op}\cdot{}^{i}\vec{V}^{op}\right]dt$$

$$= \left(\frac{1}{2}m\,{}^{i}\vec{V}^{op}\cdot{}^{i}\vec{V}^{op}\right)\Big|_{t_1}^{t_2}$$

$$= T(t_2) - T(t_1) \qquad (30)$$

Work = Change in Kinetic Energy

Particle on a String



As the particle spins around the fixed point, O, the string is shortened from r_o to r_f. If $\dot{\theta}(0) = \dot{\theta}_o$, what is $\dot{\theta}_f$?

Moment: $\vec{M}^{o} = \vec{R}^{op} \times F_s\hat{e}_r = \vec{0}$

$\Rightarrow$ we have conservation of angular momentum:

$$\vec{H}^{o} = \vec{R}^{op}\times m\,\frac{{}^{i}d\vec{R}^{op}}{dt} = r\hat{e}_r \times m(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta)$$

$$= mr^2\dot{\theta}\,\hat{e}_n = \text{constant}$$

$$\Rightarrow r^2\dot{\theta} = r_o^2\dot{\theta}_o = r_f^2\dot{\theta}_f = \text{constant}$$

$$\therefore \boxed{\dot{\theta}_f = \frac{r_o^2}{r_f^2}\,\dot{\theta}_o}$$

As $r_f \to 0$, $\dot{\theta}_f \to \infty$.

Is any work done?

The work done can be computed from the Work-Energy equation:

$$W = \Delta T = T(t_f) - T(t_o)$$

$$T = \tfrac{1}{2} m\, {}^{i}\vec{V}^{op} \cdot {}^{i}\vec{V}^{op}$$

$$= \tfrac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right)$$

Noting that $\dot{r} = 0$ @ t_o and t_f we have

$$T_o = \tfrac{1}{2} m r_o^2 \dot{\theta}_o^2$$

$$T_f = \tfrac{1}{2} m r_f^2 \dot{\theta}_f^2$$

Using $\dot{\theta}_f = \frac{r_o^2}{r_f^2} \dot{\theta}_o \Rightarrow T_f = \frac{1}{2} m r_f^2 \frac{r_o^4}{r_f^4} \dot{\theta}_o^2$

or $$T_f = \frac{1}{2} m \frac{r_o^4 \dot{\theta}_o^2}{r_f^2}$$

If $r_f < r_o \Rightarrow T_f > T_o$

$$W = T_f - T_o = \frac{1}{2} m \dot{\theta}_o^2 \left(\frac{r_o^4}{r_f^2} - r_o^2 \right)$$

$$\boxed{W = \frac{1}{2} m \dot{\theta}_o^2 r_o^2 \left(\frac{r_o^2}{r_f^2} - 1 \right)}$$