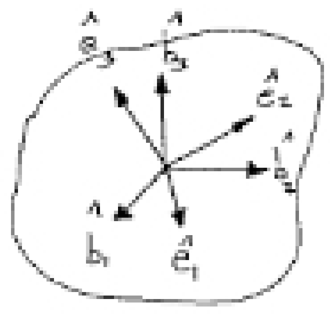
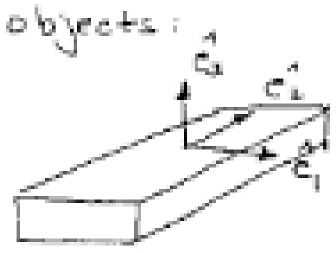
AAE 340
Lecture #31

215



b frame = arb. frame
e frame = eigenvectors along Principal Axes

Often we can see the PA directions in simple objects:



216

Summary

Eigenvalue Problem:

$$[I]\{\omega\} = I^*\{\omega\}$$

1. PMOI values I_1^*, I_2^*, I_3^* are eigenvalues.
2. PA directions $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are eigenvectors.

If I^*'s are distinct, then PA's are mutually orthogonal.

217

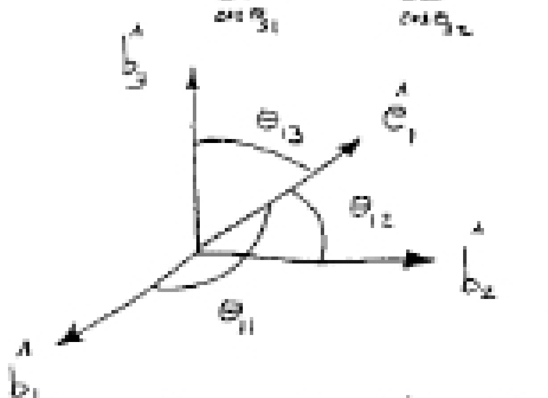
Eigenvectors

Suppose we have solved for 3 eigenvectors, $\hat{e}_j$, of Eq. 106, p. 214, $j = 1, 2, 3$.

Assume they are unit vectors.

In the b coords,

$$\hat{e}_1 = e_{11}\hat{b}_1 + e_{12}\hat{b}_2 + e_{13}\hat{b}_3$$
$$\hat{e}_2 = e_{21}\hat{b}_1 + e_{22}\hat{b}_2 + e_{23}\hat{b}_3$$
$$\hat{e}_3 = \underbrace{e_{31}}_{\cos\theta_1}\hat{b}_1 + \underbrace{e_{32}}_{\cos\theta_2}\hat{b}_2 + \underbrace{e_{33}}_{\cos\theta_3}\hat{b}_3 \qquad (107)$$



Then e_{11} is the direction cosine,

$e_{11} = \cos\theta_{11}$

between the $\hat{e}_1$ and $\hat{b}_1$ axes.

218

In the e frame we can have

$$\vec{r}\,' = r_1'\hat{e}_1 + r_2'\hat{e}_2 + r_3'\hat{e}_3 \qquad (108)$$

and in the b frame

$$\vec{r} = r_1\hat{b}_1 + r_2\hat{b}_2 + r_3\hat{b}_3 \qquad (109)$$

Setting $\vec{r}\,' = \vec{r}$ and using (107) gives

$$\{r'\} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}\{r\} \qquad (110)$$

which is the same form as Eq. 83, p. 202, so we write

$$\{r'\} = \underbrace{[\;E\;]}_{\substack{\text{direction} \\ \text{cosine matrix.} \\ \text{Rows are eigenvectors.}}}\{r\} \qquad (111)$$

Eq. 111 provides the transformation between arb. body axes (b frame) and the PA's (e frame).

In the e frame, the ang. mom. is from Eq. 89, p. 204:

$$\{H'\} = [I']\{\omega'\} \qquad (89)$$

where we know that $[I']$ is a <u>diagonal</u> matrix composed of <u>PMoI's</u> (the eigenvalues).

The $[\ell]$ matrix of Eqs. 90-92, p. 204, becomes $[E]$

$$[E] = [\ell] \qquad (112)$$

so from (92) we have

$$[I'] = \begin{bmatrix} I_1^* & 0 & 0 \\ 0 & I_2^* & 0 \\ 0 & 0 & I_3^* \end{bmatrix} = [E]\underbrace{[I]}_{\substack{\text{old}\\ \text{inertia}\\ \text{matrix}}}[E]^t \qquad (113)$$

where $[E]$ is defined by the eigenvectors (rows) in (110).

This direction cosine matrix, $[E]$, operates on the old inertia matrix through a similarity trans. to produce a diagonal matrix.

<u>Notes</u>

$$\hat{e}_1 = e_{11}\hat{b}_1 + e_{12}\hat{b}_2 + e_{13}\hat{b}_3$$
$$\hat{e}_2 = e_{21}\hat{b}_1 + e_{22}\hat{b}_2 + e_{23}\hat{b}_3$$
$$\hat{e}_3 = e_{31}\hat{b}_1 + e_{32}\hat{b}_2 + e_{33}\hat{b}_3$$

$$\underbrace{\hat{e}_1 = \begin{Bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{Bmatrix}, \quad \hat{e}_2 = \begin{Bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{Bmatrix}, \quad \hat{e}_3 = \begin{Bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{Bmatrix}}_{\text{b coords. of } \hat{e}_j}$$

$$r_1'\hat{e}_1 + r_2'\hat{e}_2 + r_3'\hat{e}_3 = r_1\hat{b}_1 + r_2\hat{b}_2 + r_3\hat{b}_3$$

$$= r_1'\begin{Bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{Bmatrix} + r_2'\begin{Bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{Bmatrix} + r_3'\begin{Bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{Bmatrix}$$

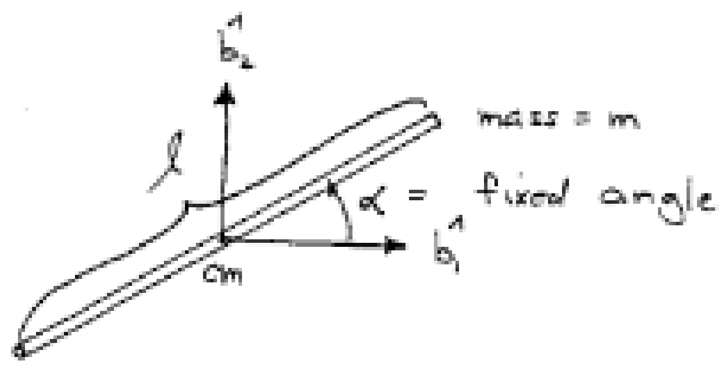
$$= \begin{bmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \end{bmatrix}\begin{Bmatrix} r_1' \\ r_2' \\ r_3' \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} r_1' \\ r_2' \\ r_3' \end{Bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix}$$

AAE 340

Lecture # 32

Eigenvalues and Eigenvectors of a Thin Rod at Fixed Angle α



Recall from Eq. 44, p. 193, that the inertia matrix in the b frame is

$$I = \frac{ml^2}{12}\begin{bmatrix} \sin^2\alpha & -\sin\alpha\cos\alpha & 0 \\ -\sin\alpha\cos\alpha & \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (a)$$

Find the eigenvalues, $\bar{I}_1^*, \bar{I}_2^*, \bar{I}_3^*$, from Eqs. (103) and (a):

$$\begin{vmatrix} \sin^2\alpha - \bar{I}^* & -\sin\alpha\cos\alpha & 0 \\ -\sin\alpha\cos\alpha & \cos^2\alpha - \bar{I}^* & 0 \\ 0 & 0 & 1-\bar{I}^* \end{vmatrix} = 0 \qquad (b)$$

$$\Rightarrow (1-\bar{I}^*)\left[(s_\alpha^2 - \bar{I}^*)(c_\alpha^2 - \bar{I}^*) - s_\alpha^2 c_\alpha^2\right] = 0 \qquad (c)$$

which implies that either

$$1 - \bar{I}^* = 0$$

$$\Rightarrow \boxed{\bar{I}_1^* = 1} \qquad (d)$$

or

$$\cancel{s_\alpha^2 c_\alpha^2} - s_\alpha^2 \bar{I}^* - \bar{I}^* c_\alpha^2 + \bar{I}^{*2} - \cancel{s_\alpha^2 c_\alpha^2} = 0 \qquad (e)$$

$$\Rightarrow \bar{I}^*\left[\bar{I}^* - 1\right] = 0 \qquad (f)$$

$$\Rightarrow \boxed{\begin{matrix} \bar{I}_2^* = 1 \\ \bar{I}_3^* = 0 \end{matrix}} \qquad (g)$$

Thus, we have a repeated root $\bar{I}_{1,2}^* = 1$.

(Note that the actual values are $I_1^* = I_2^* = \frac{ml^2}{12}\bar{I}_{1,2}^*$.)

Find the eigenvectors $\hat{e}_1, \hat{e}_2$ and $\hat{e}_3$.

First let us solve for the eigenvector corresponding to the distinct root $I_3^* = 0$, namely $\hat{e}_3$:

Then from Eqs. (102), p. 211 and (a) we have

$$\frac{ml^2}{12}\begin{bmatrix} s_\alpha^2 - 0 & -s_\alpha c_\alpha & 0 \\ -s_\alpha c_\alpha & c_\alpha^2 - 0 & 0 \\ 0 & 0 & 1-0 \end{bmatrix}\begin{Bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (h)$$

$$\Rightarrow \quad s_\alpha^2 e_{31} - s_\alpha c_\alpha e_{32} = 0 \qquad (i)$$

$$-s_\alpha c_\alpha e_{31} + c_\alpha^2 e_{32} = 0 \qquad (j)$$

$$\boxed{e_{33} = 0} \qquad (k)$$

Eq. (k) implies that the eigenvector $\hat{e}_3$ lies in the $\hat{b}_1 \hat{b}_2$ plane.

Eqs. (i) and (j) can be simplified to

$$s_\alpha e_{31} - c_\alpha e_{32} = 0 \qquad (l)$$

$$s_\alpha e_{31} - c_\alpha e_{32} = 0 \qquad (m)$$

$\Rightarrow$ Not independent. We can write

$$e_{32} = e_{31}\tan\alpha \qquad (n)$$

Since

$$e_{31}^2 + e_{32}^2 + \overset{0}{e_{33}^2} = 1 \qquad (o)$$

we have, by substituting (n) into (o):

$$e_{31}^2 + e_{31}^2\tan^2\alpha = 1$$

$$\Rightarrow \quad e_{31} = \frac{1}{\sqrt{1+\tan^2\alpha}} \qquad (p)$$

Thus from (n) and (p) we have

$$e_{32} = \frac{\tan(\alpha)}{\sqrt{1+\tan^2\alpha}} \quad \text{and}$$

$$\hat{e}_3 = \frac{(1, \tan\alpha, 0)^t}{\sqrt{1+\tan^2\alpha}} = \cos\alpha\,(1, \tan\alpha, 0)^t$$

(q)