AAE 340
Lecture #37

Exam 3

Last time we found the solutions for Euler's EOM's with arbitrary I.C.'s, for a symmetric torque-free RB:

$$\boxed{\begin{aligned} \omega_x(t) &= \omega_{x0}\cos\lambda t - \omega_{y0}\sin\lambda t \\ \omega_y(t) &= \omega_{y0}\cos\lambda t + \omega_{x0}\sin\lambda t \\ \text{where } \lambda &= \left(\frac{I_a - I_t}{I_t}\right)\Omega \qquad (180) \end{aligned}}$$

Note from Eqs. 180 that

$$\begin{aligned} \omega_x^2(t) + \omega_y^2(t) &= \omega_{x0}^2 c^2 - \cancel{2\omega_{x0}\omega_{y0} cs} + \omega_{y0}^2 s^2 \\ &\quad + \omega_{y0}^2 c^2 + \cancel{2\omega_{y0}\omega_{x0} cs} + \omega_{x0}^2 s^2 \\ &= \omega_{x0}^2 + \omega_{y0}^2 = \text{constant} \end{aligned}$$

We also note from Eq. 160 that

$$\omega_z(t) = \Omega = \text{constant}$$

Since $\quad {}^e\vec{\omega}^b = \omega_x(t)\hat{b}_1 + \omega_y(t)\hat{b}_2 + \omega_z(t)\hat{b}_3 \qquad (181)$

we see that

$$\begin{aligned} \left|{}^e\vec{\omega}^b(t)\right| &= \sqrt{{}^e\vec{\omega}^b \cdot {}^e\vec{\omega}^b} \\ &= \sqrt{\omega_{x0}^2 + \omega_{y0}^2 + \Omega^2} \\ &= \text{constant} \qquad (182) \end{aligned}$$

Thus the length of ${}^e\vec{\omega}^b(t)$ is a constant, but ${}^e\vec{\omega}^b(t)$ itself is obviously not a constant.

Define a new vector:

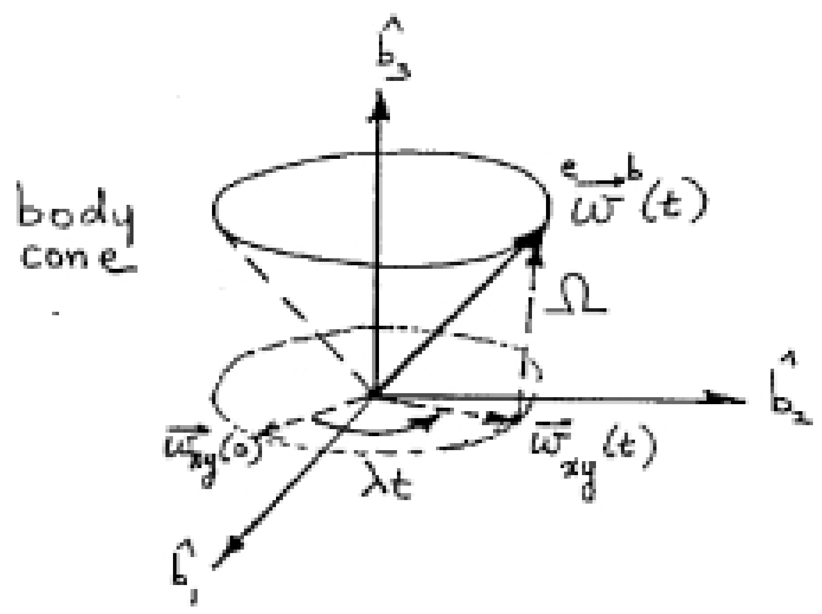
$$\vec{\omega}_{xy} \equiv \omega_x(t)\hat{b}_1 + \omega_y(t)\hat{b}_2 \qquad (183)$$

This is the projection of ${}^e\vec{\omega}^b(t)$ onto the xy body plane.

We note that

$$\left|\vec{\omega}_{xy}(t)\right| = \sqrt{\omega_{x0}^2 + \omega_{y0}^2} = \text{constant} \qquad (184)$$

Thus, with the knowledge we have gained by examining the analytic sol^ns of ω_x, ω_y and ω_z (Eqs. 160 and 180), we can now sketch the angular velocity vector ${}^e\vec{\omega}^b(t)$ in the body frame.



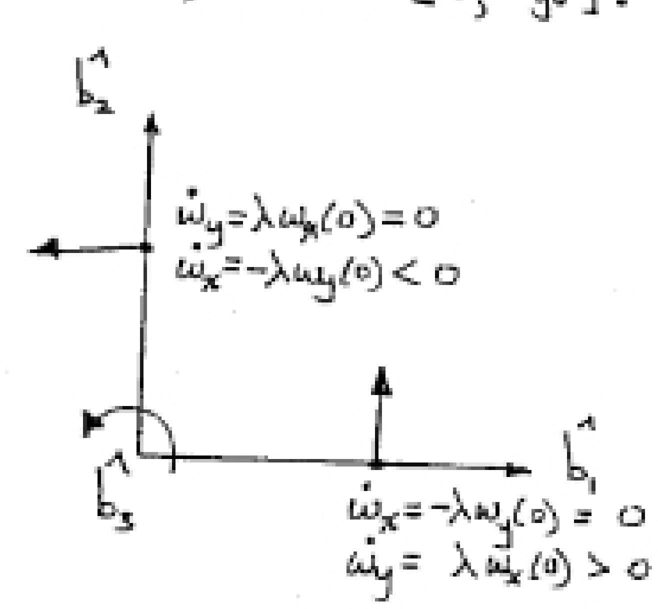
The vector $\omega_{xy}(t)$ rotates at a rate of λ radians/second about the $\hat{b}_3$ axis, and has a constant magnitude.

The total angular velocity vector, ${}^e\vec{\omega}^b(t)$, forms a body cone as it rotates around the $\hat{b}_3$ axis (at rate λ).

We can determine which way ${}^e\vec{\omega}^b$ rotates about $\hat{b}_3$ by considering the EOMs:

$$\begin{aligned} \dot{\omega}_x &= -\lambda\omega_y \\ \dot{\omega}_y &= \lambda\omega_x \end{aligned} \qquad \text{from Eqs 162 and 163 (p. 242)}$$

Assume λ = positive and consider two test points $[\omega_{x0}, 0]$ and $[0, \omega_{y0}]$:



The direction of motion is counterclockwise for $\lambda > 0 \Rightarrow$ rotation is about $+\hat{b}_3$.

Behavior of the Angular Momentum Vector

From Euler's Law

$$\vec{M}^c = {}^e\dot{\vec{H}}^c = 0$$

since there are no moments. Thus,

$${}^e\vec{H}^c = \text{constant} \qquad (185)$$

in inertial space.

But, when we view ${}^e\vec{H}^c$ in the body frame we have (from Eq. 140):

$${}^e\vec{H}^c = I_x \omega_x \hat{b}_1 + I_y \omega_y \hat{b}_2 + I_z \omega_z \hat{b}_3$$

We note that

$$\left| {}^e\vec{H}^c \right| = \sqrt{I_x^2 \omega_x^2 + I_y^2 \omega_y^2 + I_z^2 \omega_z^2}$$
$$= \text{constant}$$

since $I_x = I_y$ and $\omega_x^2 + \omega_y^2$ are constants.

We can write ${}^e\vec{H}^c$ as

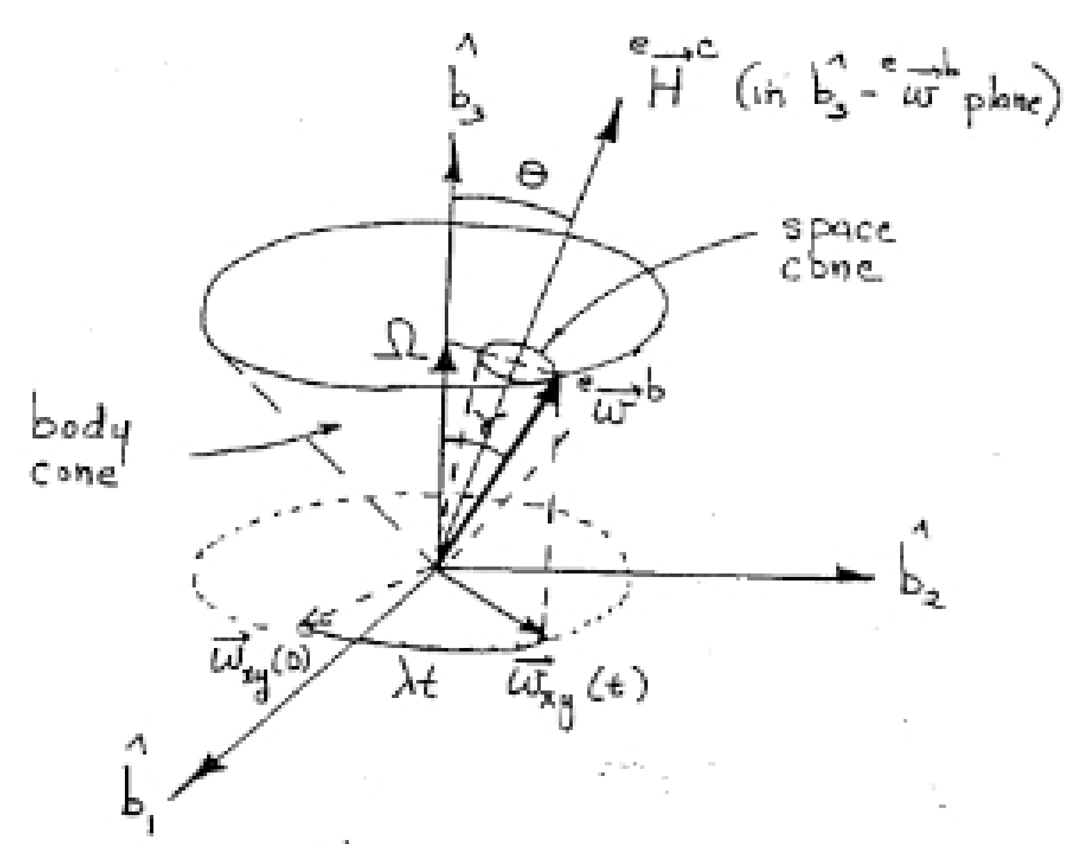
$${}^e\vec{H}^c = I_t \left(\omega_x \hat{b}_1 + \omega_y \hat{b}_2 \right) + I_a \Omega \hat{b}_3$$
$$= I_t \vec{\omega}_{xy} + I_a \Omega \hat{b}_3 \qquad (186)$$

Clearly ${}^e\vec{H}^c$ is in the plane defined by

$$\vec{\omega}_{xy} \text{ and } \hat{b}_3$$

or, the plane defined by

$${}^e\vec{\omega}^b(t) \text{ and } \hat{b}_3 .$$



Notes

γ is the angle between ${}^e\vec{\omega}^b$ and $\hat{b}_3$. It is constant:

$$\tan \gamma = \frac{|\vec{\omega}_{xy}|}{\Omega} = \text{constant} \qquad (187)$$

θ is the angle between ${}^e\vec{H}^c$ and $\hat{b}_3$. It is also constant:

$$\tan \theta = \frac{I_t |\vec{\omega}_{xy}|}{I_a \Omega} = \text{constant} \qquad (188)$$

From Eqs. 187 and 188:

$$\tan \theta = \frac{I_t}{I_a} \tan \gamma \qquad (189)$$

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CHAPTER 8

EULERIAN ANGLES

A rigid body has 3 DOF's (Degrees of Freedom) in orientation.

Knowledge of the angular velocities $\omega_x(t)$, $\omega_y(t)$ and $\omega_z(t)$ does not provide the orientation of the b frame wrt the e frame.

Integration of the angular velocities, $\int \omega_x(t)\,dt$, $\int \omega_y(t)\,dt$, and $\int \omega_z(t)\,dt$, results in angles which do not provide the orientation.

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Euler realized that if three angles of rotation about each body axis $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$ were used, then the order of rotation makes a difference.

Proof (By Demonstration):

Rotate an eraser by 90° around its x, y and z axes.

Consider two Euler Angle Sequences:

1-2-3 Sequence:

1. Rotate around x by $+90°$
2. Rotate around y by $+90°$
3. Rotate around z by $+90°$

3-2-1 Sequence:

1. Rotate around z by $+90°$
2. Rotate around y by $+90°$
3. Rotate around x by $+90°$

1-2-3 Sequence

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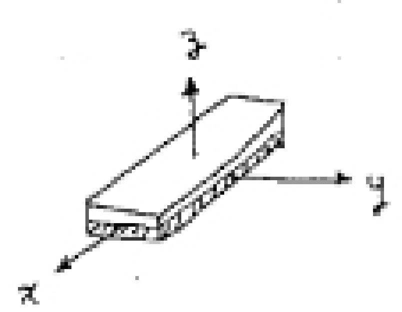
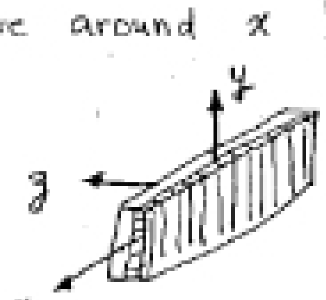
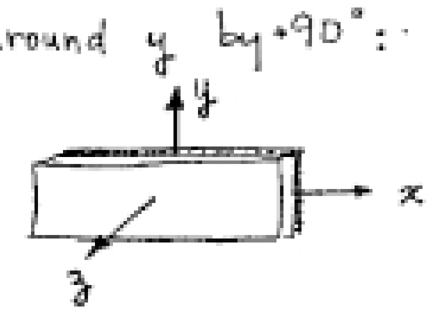


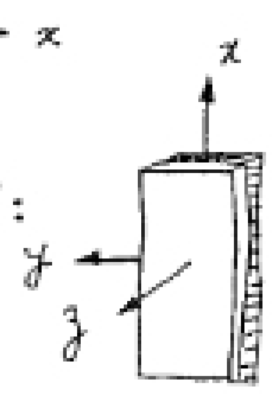
1. Rotate around x by $+90°$:



2. Rotate around y by $+90°$:



3. Rotate around z by $+90°$:



Note: x pointing up.

3-2-1 Sequence

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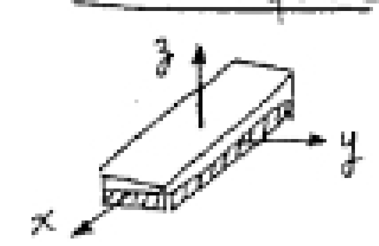
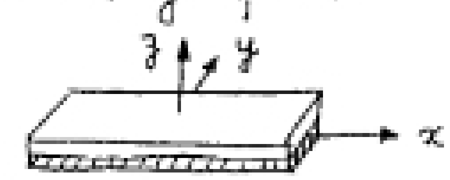
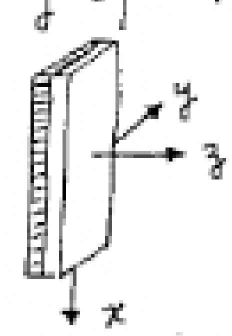


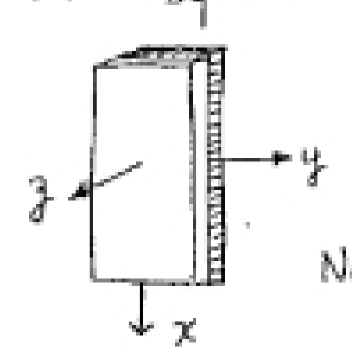
1. Rotate around z by $+90°$



2. Rotate around y by $+90°$



3. Rotate around x by $+90°$



Note: x pointing down.