AAE 340
Lecture # 25

Skip Lectures 23 and 24, pp. 169–177.

Exam 2

CHAPTER 7 RIGID BODY DYNAMICS

Involves 2 Major Parts:

1. RB Kinematics (Dimensions L, T)

   The kinematics of a RB involves 6 DOF's (Degrees of Freedom):

   3 Translational (Position of C.M)

   3 Rotational (Orientation of Body in Space)

2. RB EOM's (Eq^{ns} of Motion)

   Translational: $\vec{F} = m\,{}^{e}\ddot{\vec{r}}^{\,oc}$ (1)

   Rotational: $\vec{M}^{q} = {}^{e}\dot{\vec{H}}^{q}$ (2)

   where ref. pt. q is 1. Fixed or 2. Center of Mass.

If $\vec{F}$ is independent of the rotational motion and if $\vec{M}^{q}$ is independent of the center of mass,
then: Eqs. (1) and (2) can be solved separately.

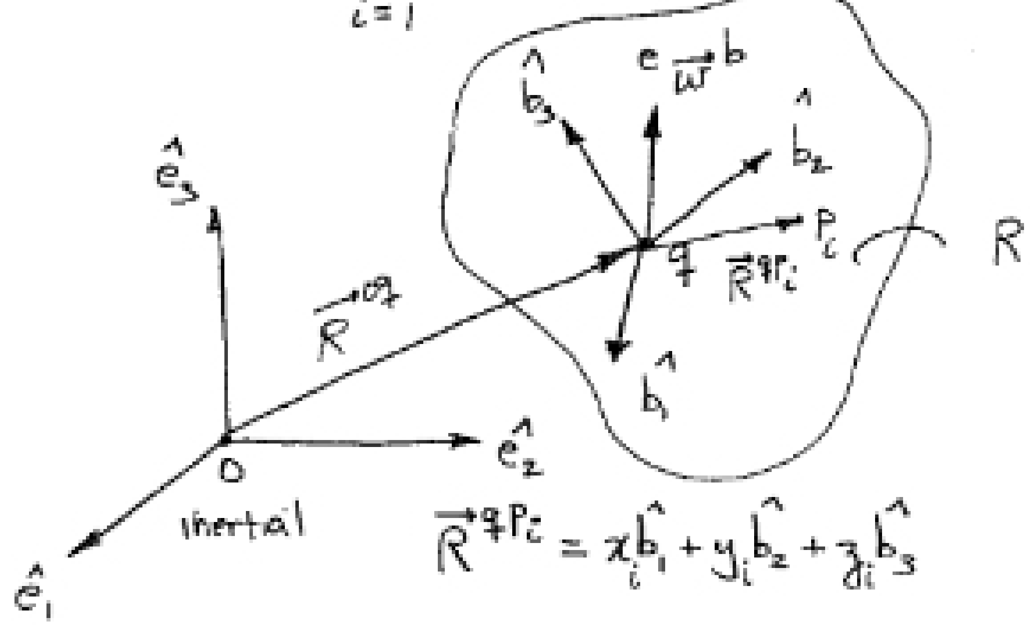
In Chap. 7 we discuss RB EOM's.
In Chap. 8 we discuss RB Kinematics.

Skip pp. 180 and 181.

Moments of Inertia (MOI's) for RB and Products of Inertia (POI's)

From system of particles, the ang. mom. relative to q is (Eq. 14, p. 145)

$$\vec{H}^{q} = \sum_{i=1}^{n} \vec{R}^{qP_i} \times m_i\,{}^{e}\dot{\vec{R}}^{qP_i} \qquad (3)$$



$$\vec{R}^{qP_i} = x_i\hat{b}_1 + y_i\hat{b}_2 + z_i\hat{b}_3$$

For a rigid body:

$${}^{e}\dot{\vec{R}}^{qP_i} = \cancel{{}^{b}\frac{d}{dt}\left(\vec{R}^{qP_i}\right)} + {}^{e}\vec{\omega}^{b} \times \vec{R}^{qP_i} \qquad (4)$$

0 because P_i do not move wrt q in RB (b frame).

From (3) and (4)

$$\vec{H}^{q} = \sum_{i=1}^{n} m_i\,\vec{R}^{qP_i} \times \left({}^{e}\vec{\omega}^{b} \times \vec{R}^{qP_i}\right) \qquad (5)$$

Replace m_i with $\rho\, dV$

ρ = mass density, dV = volume

then Eq. 5 becomes

$$\vec{H}^{q} = \int_V \rho\, \vec{R}^{qP} \times \left({}^{e}\vec{\omega}^{b} \times \vec{R}^{qP}\right) dV \qquad (6)$$

Since the body axis system is fixed in the RB with origin @ q, let

$$\vec{R}^{qP} = x\hat{b}_1 + y\hat{b}_2 + z\hat{b}_3 \qquad (7)$$

$${}^{e}\vec{\omega}^{b} = \omega_x\hat{b}_1 + \omega_y\hat{b}_2 + \omega_z\hat{b}_3 \qquad (8)$$

$$\vec{R}^{qP} \times \left({}^{e}\vec{\omega}^{b} \times \vec{R}^{qP} \right) =$$

$$\left[(y^2+z^2)\omega_x - xy\,\omega_y - xz\,\omega_z \right] \hat{b}_1$$
$$+ \left[-yx\,\omega_x + (x^2+z^2)\omega_y - yz\,\omega_z \right] \hat{b}_2$$
$$+ \left[-zx\,\omega_x - zy\,\omega_y + (x^2+y^2)\omega_z \right] \hat{b}_3 \qquad (9)$$

Multiply Eq. 9 by $\rho\, dV$ (or dm) and integrate over body to obtain Eq. 6.

Define the <u>MOI's</u> (Moments of Inertia)

$$I_{xx} = \int_V \rho\,(y^2+z^2)\,dV \qquad (10)$$

$$I_{yy} = \int_V \rho\,(x^2+z^2)\,dV \qquad (11)$$

$$I_{zz} = \int_V \rho\,(x^2+y^2)\,dV \qquad (12)$$

Define the <u>POI's</u> (Products of Inertia)

$$I_{xy} = \int_V -xy\,\rho\,dV = I_{yx} \qquad (13)$$

$$I_{xz} = \int_V -xz\,\rho\,dV = I_{zx} \qquad (14)$$

$$I_{yz} = \int -yz\,\rho\,dV = I_{zy} \qquad (15)$$

Using (10)-(15) in (6) we obtain

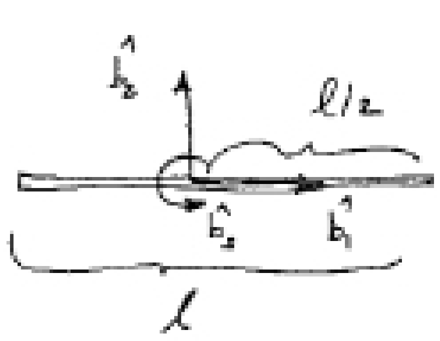
$$\vec{H}^{q} = \left(I_{xx}\,\omega_x + I_{xy}\,\omega_y + I_{xz}\,\omega_z \right) \hat{b}_1$$
$$+ \left(I_{yx}\,\omega_x + I_{yy}\,\omega_y + I_{yz}\,\omega_z \right) \hat{b}_2$$
$$+ \left(I_{zx}\,\omega_x + I_{zy}\,\omega_y + I_{zz}\,\omega_z \right) \hat{b}_3 \qquad (16)$$

Usually q is taken @ the c.m. However, this is not req'd in the def^n.

In Matrix Notation, Eq. 16 is

$$\underset{3\times 1}{\{H\}} = \underset{3\times 3}{\left[\; I \;\right]} \underset{3\times 1}{\{\omega\}} \qquad (17)$$

<u>Example: MOI's of Thin Rod</u>



Mass $= m$, $\rho = \dfrac{m}{l}$ (linear density)

From (10), p. 184:

$$I_{xx} = \int_{-l/2}^{l/2} \frac{m}{l}\,(y^2+z^2)\,dx = 0 \qquad (18)$$

$$I_{yy} = \int_{-l/2}^{l/2} \frac{m}{l}\,(x^2+z^2)\,dx = \frac{m}{l}\,\frac{x^3}{3}\Big|_{-l/2}^{l/2}$$

$$= \frac{m}{l}\,\frac{1}{3}\left[\frac{l^3}{8} - -\frac{l^3}{8} \right] = \frac{m l^2}{12} \qquad (19)$$

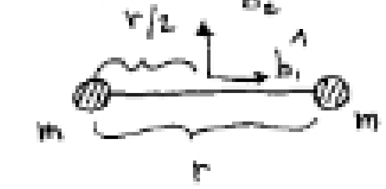
$$I_{zz} = \int_{-l/2}^{l/2} \frac{m}{l}\,(x^2+y^2)\,dx = \frac{m l^2}{12} \qquad (20)$$

We note that the POI's are all zero, from Eq^{ns} 13 - 15, p. 185:

$$I_{xy} = I_{xz} = I_{yz} = 0 \qquad (21)$$

Because each integral contains xy, xz or yz and $y = z = 0$.

<u>Example</u>: Dumbbell Satellite



<u>MOI's</u>

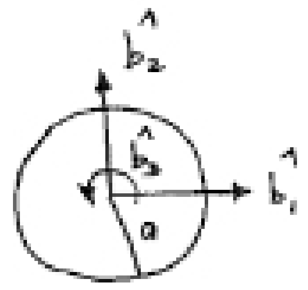
$$I_{xx} = \int_V \rho\,(y^2+z^2)\,dV = 0 \qquad (22)$$

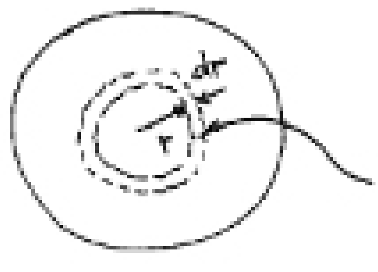
$$I_{yy} = \int_V \rho\,(x^2+z^2)\,dV = m\left(\frac{r}{2}\right)^2 + m\left(\frac{-r}{2}\right)^2$$
$$= \frac{m r^2}{2} \qquad (23)$$

$$I_{zz} = \int_V \rho\,(x^2+y^2)\,dV = \frac{m r^2}{2} \qquad (24)$$

Again the POI's are all zero, because Eq^ns (13)-(15) have xy, xz or yz in the integrals.

Example: MOI's of Thin Disk





Replace ρdV by dm

$$dm = \underbrace{2\pi r\, dr}_{\text{area of ring}} \frac{m}{\pi a^2} \Big\} \text{area of disk} \qquad (25)$$

Thus

$$I_{zz} = \int_V \rho(\overbrace{x^2+y^2}^{r^2})\, dV = \int_0^a \frac{2\pi r^3}{\pi a^2} m\, dr$$

$$= \frac{\cancel{2}m}{a^2} \frac{r^4}{\cancel{4}_{2}} \Big|_0^a = \frac{ma^2}{2} \qquad (26)$$

Note that we have a plane figure and so

$$z = 0 \qquad (27)$$

Thus, in this case (and all similar cases):

$$I_{xx} = \int_V \rho\,(y^2+z^2)\, dV = \int_V \rho\, y^2\, dV \qquad (28)$$

$$I_{yy} = \int_V \rho\,(x^2+z^2)\, dV = \int_V \rho\, x^2\, dV \qquad (29)$$

And,

$$I_{xx} + I_{yy} = \int_V \rho\,(x^2+y^2)\, dV = I_{zz} \qquad (30)$$

The MOI of a Plane Body about a Perpendicular Axis is Equal to the Sum of the MOI's about Orthogonal Axes in the Plane (for a common ref. pt.).

Thus, since we have symmetry, for a disk

$$I_{xx} = I_{yy} \qquad (31)$$

and

$$I_{xx} = \tfrac{1}{2} I_{zz} = \tfrac{1}{4} ma^2 \qquad (32)$$

$$I_{yy} = \tfrac{1}{2} I_{zz} = \tfrac{1}{4} ma^2 \qquad (33)$$