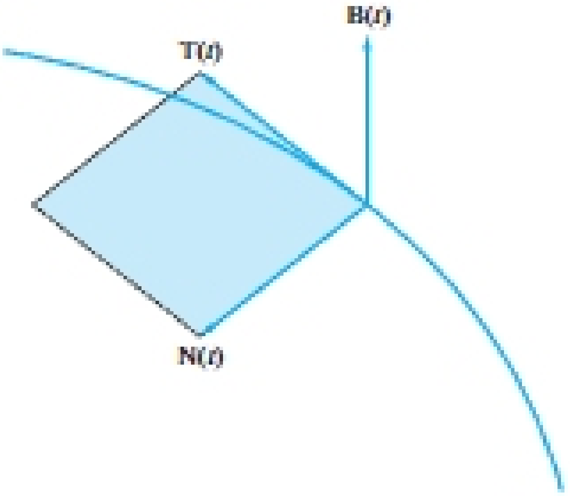
CHAPTER 2



Determinants

With each square matrix, it is possible to associate a real number called the determinant of the matrix. The value of this number will tell us whether the matrix is singular.

In Section 1, the definition of the determinant of a matrix is given. In Section 2, we study properties of determinants and derive an elimination method for evaluating determinants. The elimination method is generally the simplest method to use for evaluating the determinant of an $n \times n$ matrix when $n > 3$. In Section 3, we see how determinants can be applied to solving $n \times n$ linear systems and how they can be used to calculate the inverse of a matrix. Applications of determinants to cryptography and to Newtonian mechanics are also presented in Section 3. Further applications of determinants are presented in Chapters 3 and 6.

2.1 The Determinant of a Matrix

With each $n \times n$ matrix A, it is possible to associate a scalar, $\det(A)$, whose value will tell us whether the matrix is nonsingular. Before proceeding to the general definition, let us consider the following cases:

Case 1. 1 x 1 Matrices If $A = (a)$ is a 1×1 matrix, then A will have a multiplicative inverse if and only if $a \neq 0$. Thus, if we define

$$\det(A) = a$$

then A will be nonsingular if and only if $\det(A) \neq 0$.

Case 2. 2 x 2 Matrices Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

By Theorem 1.5.2, A will be nonsingular if and only if it is row equivalent to I. Then, if $a_{11} \neq 0$, we can test whether A is row equivalent to I by performing the following operations:

1. Multiply the second row of A by a_{11}

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{11}a_{21} & a_{11}a_{22} \end{bmatrix}$$

2. Subtract a_{21} times the first row from the new second row

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{11}a_{22} - a_{21}a_{12} \end{bmatrix}$$

Since $a_{11} \neq 0$, the resulting matrix will be row equivalent to I if and only if

$$a_{11}a_{22} - a_{21}a_{12} \neq 0 \tag{1}$$

If $a_{11} = 0$, we can switch the two rows of A. The resulting matrix

$$\begin{bmatrix} a_{21} & a_{22} \\ 0 & a_{12} \end{bmatrix}$$

will be row equivalent to I if and only if $a_{21}a_{12} \neq 0$. This requirement is equivalent to condition (1) when $a_{11} = 0$. Thus, if A is any 2×2 matrix and we define

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

then A is nonsingular if and only if $\det(A) \neq 0$.

Notation

We can refer to the determinant of a specific matrix by enclosing the array between vertical lines. For example, if

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

then

$$\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix}$$

represents the determinant of A.

Case 3. 3×3 Matrices We can test whether a 3×3 matrix is nonsingular by performing row operations to see if the matrix is row equivalent to the identity matrix I. To carry out the elimination in the first column of an arbitrary 3×3 matrix A, let us first assume that $a_{11} \neq 0$. The elimination can then be performed by subtracting a_{21}/a_{11} times the first row from the second and a_{31}/a_{11} times the first row from the third:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \dfrac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \dfrac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \\ 0 & \dfrac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}} & \dfrac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} \end{bmatrix}$$

The matrix on the right will be row equivalent to I if and only if

$$a_{11}\begin{vmatrix} \dfrac{a_{11}a_{22}-a_{21}a_{12}}{a_{11}} & \dfrac{a_{11}a_{23}-a_{21}a_{13}}{a_{11}} \\ \dfrac{a_{11}a_{32}-a_{31}a_{12}}{a_{11}} & \dfrac{a_{11}a_{33}-a_{31}a_{13}}{a_{11}} \end{vmatrix} \neq 0$$

Although the algebra is somewhat messy, this condition can be simplified to

$$\begin{aligned} a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} \\ + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \neq 0 \end{aligned} \tag{2}$$

Thus, if we define

$$\begin{aligned} \det(A) = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} \\ + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \end{aligned} \tag{3}$$

then, for the case $a_{11} \neq 0$, the matrix will be nonsingular if and only if $\det(A) \neq 0$.

What if $a_{11} = 0$? Consider the following possibilities:

(i) $a_{11} = 0, a_{21} \neq 0$
(ii) $a_{11} = a_{21} = 0, a_{31} \neq 0$
(iii) $a_{11} = a_{21} = a_{31} = 0$

In case (i), it is not difficult to show that A is row equivalent to I if and only if

$$-a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \neq 0$$

But this condition is the same as condition (2) with $a_{11} = 0$. The details of case (i) are left as an exercise for the reader (see Exercise 7 at the end of this section).

In case (ii), it follows that

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is row equivalent to I if and only if

$$a_{31}(a_{12}a_{23} - a_{22}a_{13}) \neq 0$$

Again, this is a special case of condition (2) with $a_{11} = a_{21} = 0$.

Clearly, in case (iii) the matrix A cannot be row equivalent to I and hence must be singular. In this case, if we set a_{11}, a_{21}, and a_{31} equal to 0 in formula (3), the result will be $\det(A) = 0$.

In general, then, formula (2) gives a necessary and sufficient condition for a 3×3 matrix A to be nonsingular (regardless of the value of a_{11}).

We would now like to define the determinant of an $n \times n$ matrix. To see how to do this, note that the determinant of a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$