

LESSON #15: Negation rules (6.3) Read pp. 156-163. Homework: 6.7-6.14

POWERPOINT SLIDE #1

We have studied seven proof rules so far (and you will need to know how to use all of them in working actual proofs for the midterm exam).

POWERPOINT SLIDE #2

We've already discussed the first major proof 'method', **proof by cases** (formally: **disjunction elimination**). Recall that disjunction elimination is used to prove a conclusion that can be shown to follow separately (in separate subproofs) from each disjunct of a known disjunction. (See powerpoint slide 2 for details)

POWERPOINT SLIDE #3

Now we come to our *second* major method of proof (and another rule that requires *subproofs*): **proof by contradiction**.

This rule is based on the *one remaining Boolean connective*: **negation**. The rule's formal name is **Negation Introduction (\neg Intro)**

To understand proof by contradiction, however, you first need to know what a 'contradiction' is. The book gives this definition (p. 138): "*a contradiction is any claim that cannot possibly be true, or any set of claims which cannot all be true simultaneously.*"

POWERPOINT SLIDE #4

Contradiction symbol: \perp

The following are examples of single-sentence contradictions:

$a \neq a$
 $P \wedge \neg P$

Pairs (or larger sets) of sentences can also be contradictions (i.e., they can't both/all be true in the same world):

P \dots $\neg P$
 $A \wedge B$ \dots $\neg(A \vee B)$
Larger(a,b) \dots Larger(b,a)

POWERPOINT SLIDE #5

Now, the basic idea of proof by contradiction (i.e., \neg Intro) is to assume as a premise the *opposite* (i.e., the *negation*) of what you want to actually prove, and when that assumption leads to a *contradiction* (i.e., a result that can't possibly be true), then you can safely conclude that *its opposite must instead be true* (i.e., this opposite claim being the one you originally wanted to prove anyway).

POWERPOINT SLIDE #6

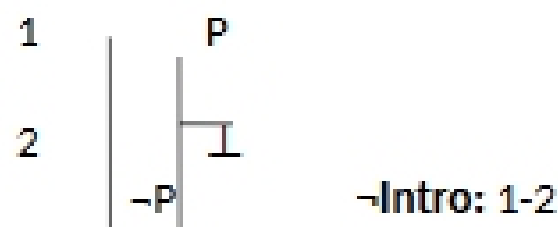
We use this kind of reasoning all the time in **everyday life**.

(Elaborate on the story of Tony and the goldfish ...)

POWERPOINT SLIDE #7

Now, we are ready to *formalize* this method of reasoning:

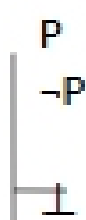
Negation Introduction (\neg Intro)



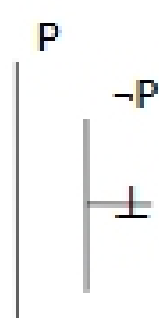
That is, to prove $\neg P$, first assume its opposite, P , as the premise of a subproof and show that within this subproof you (eventually) reach a **contradiction** (it may take any number of steps to actually reach the contradiction—I've left these intermediate steps out in the schema above). If you can prove that a contradiction follows from assuming P as the premise of the subproof in this way, then you are allowed to assert the *negation* (opposite) of P back one level within the main proof.

POWERPOINT SLIDE #8

This approach means that we also need some way of formally identifying and marking a contradiction, and we achieve this through a simple rule of Contradiction Introduction (\perp Intro), which allows you to assert \perp (i.e., that a contradiction has been reached) anytime you can cite two contradictory steps of the proof in support of this assertion.



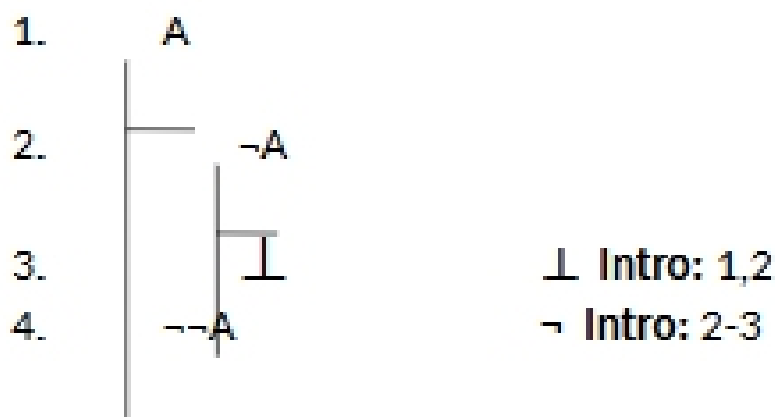
Actually, more commonly the contradiction will be part of a structure like this:



That is, you'll most often use **Contradiction Intro** *inside a subproof*, often as part of a larger strategy of *proof by contradiction*.

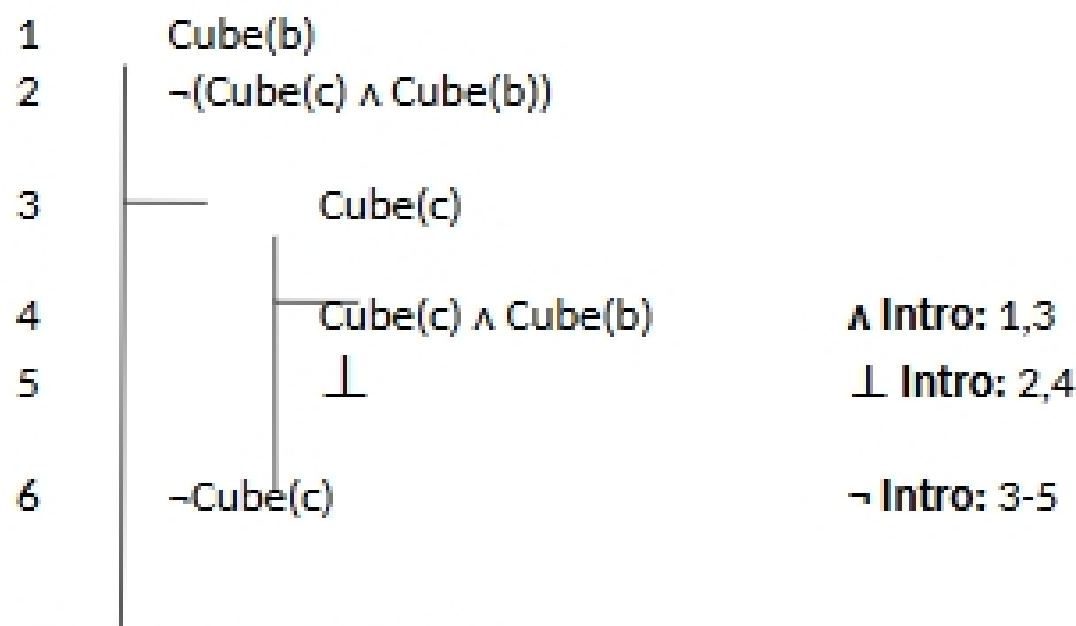
POWERPOINT SLIDE #9

Page 158 of the textbook gives a very simple example of how this works—in this case to prove that $\neg\neg A$ is a logical consequence of A . Notice how the two rules \neg **Intro** and **Contradiction Intro** are used together in the the proof by contradiction:



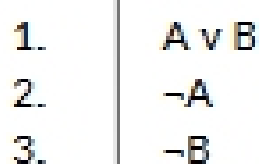
Proof 6.9 from your homework offers another good example of how \perp **Intro** is used as part of a \neg **Intro** strategy:

POWERPOINT SLIDE #10



POWERPOINT SLIDE #11

Keep in mind that \perp **Intro** doesn't always have to be used. Another possible combination of strategies is to use \vee **Elim**. The following example from p. 159 of the textbook, which is *not* contradictory:



Remember how \vee **Elim** works: You must have a **disjunction** to work from (in this case, at step 1) and then you must open **subproofs** corresponding to each of the disjuncts, where each subproof's **premise** is one of the **disjuncts** of the original disjunction (notice that the premise at step 4 is the first disjunct from step 1, and the premise at step 6 is the other disjunct from step 1). Then, if you can prove the **same conclusion in both of the subproofs**, you can assert that conclusion one level back in the main proof. In this case, both subproofs lead to a *contradiction* (steps 5 and 7), so you can assert a *contradiction* back at the main level (step 8).