LESSON #15: Negation rules (6.3) Read pp. 156-163. Homework: 6.7-6.14

POWERPOINT SLIDE #1

We have studied seven proof rules so far (and you will need to know how to use all of them in working actual proofs for the midterm exam).

POWERPOINT SLIDE #2

We've already discussed the first major proof 'method', **proof by cases** (formally: **disjunction elimination**). Recall that disjunction elimination is used to prove a conclusion that can be shown to follow separately (in separate subproofs) from each disjunct of a known disjunction. (See powerpoint slide 2 for details)

POWERPOINT SLIDE #3

Now we come to our *second* major method of proof (and another rule that requires *subproofs*): **proof by contradiction**.

This rule is based on the *one remaining Boolean connective*: **negation**. The rule's formal name is **Negation Introduction (¬ Intro)**

To understand proof by contradiction, however, you first need to know what a 'contradiction' is. The book gives this definition (p. 138): *"a contradiction is any claim that cannot possibly be true, or any set of claims which cannot all be true simultaneously."*

POWERPOINT SLIDE #4

Contradiction symbol: $\perp$

The following are examples of single-sentence contradictions:

$a \neq a$
$P \wedge \neg P$

Pairs (or larger sets) of sentences can also be contradictions (i.e., they can't both/all be true in the same world):

P	. . . .	$\neg P$
$A \wedge B$	. . . .	$\neg(A \vee B)$
Larger(a,b)	. . . .	Larger(b,a)

POWERPOINT SLIDE #5

Now, **the basic idea of <u>proof by contradiction</u>** (i.e., **¬ Intro**) is to assume as a premise the ***opposite*** (i.e., the **negation**) of what you want to actually prove, and when that assumption leads to a *contradiction* (i.e., a result that can't possibly be true), then you can safely conclude that *its opposite must instead be true* (i.e., this opposite claim being the one you originally wanted to prove anyway).

POWERPOINT SLIDE #6

We use this kind of reasoning all the time in **everyday life**.

(Elaborate on the story of Tony and the goldfish ...)

POWERPOINT SLIDE #7

Now, we are ready to ***formalize*** this method of reasoning:
Negation Introduction (¬ Intro)

```
1   |  | P
    |  |---
2   |  | ⊥
    | ¬P          ¬Intro: 1-2
```

That is, to prove **¬P**, first assume its opposite, **P**, as the premise of a subproof and show that within this subproof you (eventually) reach a **contradiction** (it may take any number of steps to actually reach the contradiction—I've left these intermediate steps out in the schema above). If you can prove that a contradiction follows from assuming P as the premise of the subproof in this way, then you are allowed to assert the *negation* (opposite) of P back one level within the main proof.

POWERPOINT SLIDE #8

This approach means that we also need some way of formally identifying and marking a contradiction, and we achieve this through a simple rule of **<u>Contradiction Introduction</u> (⊥ Intro)**, which allows you to assert ⊥ (i.e., that a contradiction has been reached) anytime you can cite two contradictory steps of the proof in support of this assertion.

```
  | P
  | ¬P
  |---
  | ⊥
```

Actually, more commonly the contradiction will be part of a structure like this:

```
| P
|  | ¬P
|  |---
|  | ⊥
|  |
```

That is, you'll most often use **Contradiction Intro** *inside a subproof*, often as part of a larger strategy of *proof by contradiction*.

POWERPOINT SLIDE #9

Page 158 of the textbook gives a very simple example of how this works—in this case to prove that **¬¬A** is a logical consequence of **A.** Notice how the two rules **¬ Intro** and **Contradiction Intro** are used together in the the proof by contradiction:

1. A
2. ¬A
3. ⊥ — **⊥ Intro:** 1,2
4. ¬¬A — **¬ Intro:** 2-3

Proof 6.9 from your homework offers another good example of how **⊥ Intro** is used as part of a **¬ Intro** strategy:

POWERPOINT SLIDE #10

1 Cube(b)
2 ¬(Cube(c) ∧ Cube(b))
3 Cube(c)
4 Cube(c) ∧ Cube(b) — **∧ Intro:** 1,3
5 ⊥ — **⊥ Intro:** 2,4
6 ¬Cube(c) — **¬ Intro:** 3-5

POWERPOINT SLIDE #11

Keep in mind that ⊥ **Intro** doesn't always have t
Another possible combination of strategies is to
following example from p. 159 of the textbook, v e
contradictory:

1. A ∨ B
2. ¬A
3. ¬B

Remember how **v Elim** works: You must have a **disjunction** to work from (in this case, at step 1) and then you must open **subproofs** corresponding to each of the disjuncts, where each subproof's **premise** is one of the **disjuncts** of the original disjunction (notice that the premise at step 4 is the first disjunct from step 1, and the premise at step 6 is the other disjunct from step 1). Then, if you can prove the ***same* conclusion in *both* of the subproofs**, you can assert that conclusion one level back in the main proof. In this case, both subproofs lead to a *contradiction* (steps 5 and 7), so you can assert a *contradiction* back at the main level (step 8).