Math 19. Lecture 29
Fast and Slow

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1 Modeling the Motion of the Heart

Suppose that we have a system of equations with a periodic solution. What can we say about the speed of the movement? For example, suppose that we look at a muscle controlling a heart valve. The muscle spends most of its time moving slightly or very slowly near one of two positions (the value is in the open position or the closed position). However, there is a rapid motion whenever the valve is opened or closed. We can use the system

$$\frac{dx}{dt} = -\frac{x^3}{3} + x + \alpha \qquad (1)$$

$$\frac{d\alpha}{dt} = -\epsilon x, \qquad (2)$$

as a simple model, where

$$\begin{aligned} x(t) &= \text{the position of the muscle at time } t, \\ \alpha(t) &= \text{the concentration of some chemical stimulus} \\ & \text{above or below a fixed concentration at time } t. \end{aligned}$$

Note that the x variable will be bounded. A heart valve can only move so far. The inverse of the parameter $\epsilon > 0$ will be used to estimate the amount of time that x spends at one or the other rest positions.[1]

[1] This equation is called van der Pol's equation and occurs in circuit theory.

2 Fast and Slow Subsystems

In equations (1) and (2), we have a slow moving system interacting with a fast moving system. If $\epsilon > 0$ is small and x is bounded, let us say that $|x| < 10$, then $d\alpha/dt$ is relatively small and cannot change quickly. Therefore, the slow moving system is

$$\frac{d\alpha}{dt} = -\epsilon x.$$

In this case, the speed at which α changes is never more than 10ϵ. On the other hand, the equation

$$\frac{dx}{dt} = -\frac{x^3}{3} + x + \alpha,$$

has no small number ϵ, and, thus, its solutions can move relatively quickly. In fact,

$$\frac{dx}{dt} = x$$

is exponential. In this case, the speed at which x can change can be relatively high.

3 What's Going On

Let us see what happens to equation

$$\frac{dx}{dt} = -\frac{x^3}{3} + x + \alpha,$$

as we change α.

- First, suppose that $\alpha = 2/3$. Then $x = 2$ is a stable equilibrium for

$$\frac{dx}{dt} = -\frac{x^3}{3} + x + \frac{2}{3}.$$

- However, α will not stay at $2/3$, since it changes according to

$$\frac{d\alpha}{dt} = -\epsilon x.$$

  This equation tells us that α will decrease slowly.

- As α decreases (but stays above $-2/3$), the equation

$$\frac{dx}{dt} = -\frac{x^3}{3} + x + \alpha \tag{3}$$

has a stable equilibrium point, say x_α, for some positive x. This equilibrium point moves slowly to the left since α changes slowly.

- At $\alpha = -2/3$, the equilibrium point becomes unstable.

- When $\alpha(t) < -2/3$, the equation (3) no longer even has a positive equilibrium point.

- As the positive equilibrium point becomes unstable and disappears, $x(t)$ must travel from where it is nearly 1 to the remaining stable equilibrium point, where $x < -2$. The motion is governed by

$$\frac{dx}{dt} = -\frac{x^3}{3} + x + \alpha$$

and so it must occur at a speed governed by this equation. Since the equation does not include the parameter ϵ, the transition from where $x \approx 1$ to where $x < -2$ should occur much faster than it took for x to decrease from 2 to 1.

- When x has switched to nearly -2, then

$$\frac{d\alpha}{dt} = -\epsilon x.$$

tells us that α must begin increasing again ($d\alpha/dt > 0$ in this case). This will be a slow increase since ϵ is small. The increase will continue as long as $\alpha < 2/3$.

- As α reaches $2/3$, the point $x = -1$ is no longer a stable equilibrium point. Note that this is the mirror image of what happened before

4 Existence of a Cycle

We can use the Poincaré-Bendixson Theorem to show that we have a periodic solution. The origin is a repelling equilibrium point, and we can find a basin