MA123, Chapter 3: The idea of limits (pp. 47-67, Gootman)

`Chapter Goals:`

- Evaluate limits.
- Evaluate one-sided limits.
- Understand the concepts of continuity and differentiability and their relationship.

`Assignments:` `Assignment 04` `Assignment 05`

Earlier, the idea of limits came up naturally in the course of defining the derivative of a function at a point. We now study limits more systematically. *Computing a limit means computing what happens to the value of a function as the variable in the expression gets closer and closer to (but does not equal) a particular value.*

▶ **The basic definition of limit:** Let f be a function of x. The expression

$$\lim_{x\to c} f(x) = L$$

means that as x gets closer and closer to c, through values both smaller and larger than c, but not equal to c, then the values of $f(x)$ get closer and closer to the value L.

Note: It may sometimes happen that the limit does not exist.

Example 1 (a): Use the tables to help evaluate $\displaystyle\lim_{x\to 2} \frac{x^2+8}{x+2}$.

x gets close to 2 from the left				
x	1.8	1.9	1.99	1.999
$\frac{x^2+8}{x+2}$				

x gets close to 2 from the right				
2.001	2.01	2.1	2.2	x
				$\frac{x^2+8}{x+2}$

Example 1 (b): Suppose that, instead of calculating all the values in the above tables, you simply substitute the value $x = 2$ into $\dfrac{x^2+8}{x+2}$. What do you find?

Note: The method of **substituting in** the limiting value of the variable works because the operations of arithmetic, namely, addition, subtraction, multiplication, and division, all behave reasonably with respect to the idea of 'getting closer to' as long as nothing illegal happens. The one illegality you will mainly have to watch out for is 'division by zero'. More precisely, if f and g are two functions one has:

$$\lim_{x\to c}\big(f(x)+g(x)\big) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \qquad \lim_{x\to c}\big(f(x)-g(x)\big) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$$

$$\lim_{x\to c}\big(f(x)\cdot g(x)\big) = \left(\lim_{x\to c} f(x)\right)\cdot\left(\lim_{x\to c} g(x)\right) \qquad \lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

as long as $\displaystyle\lim_{x\to c} g(x) \neq 0$

Example 2: Compute $\lim_{x\to 1} \left((x^2+4x+3)\cdot(2x-4)\right)$.

Example 3: Compute $\lim_{x\to 1} \dfrac{x^2-2x+1}{x+1}$.

Example 4: Suppose $\lim_{x\to 3} f(x) = -2$ and $\lim_{x\to 3} g(x) = 4$. Determine

$$\lim_{x\to 3}\left((x+1)\cdot f(x)^2 + \frac{x+2}{g(x)}\right).$$

► **Some complications with the definition of limits:** The previous examples seem to imply that "computing a limit" is the same thing as "evaluating a function". This is only true if the function in the limit is "nice enough" ("nice enough" will be defined more precisely in a few pages).

The next few examples will illustrate that the computation of $\lim_{x\to c} f(x)$ does not always reduce to the mere substitution of the value of c in place of x in the expression defining $f(x)$. The 'unusual' functions described in what follows are introduced to emphasize the fact that the notion of limit really involves what happens to the values of $f(x)$ as x gets closer to the fixed value c, and not what the value of $f(x)$ at $x=c$ is. In addition, the most interesting limits generally arise precisely when substitution gives an illegal expression involving division by 0, or even an expression of the form $\frac{0}{0}$. The latter case occurs for example when computing the derivative of a function.

► **How can a limit fail to exist?** There are two basic ways that a limit can fail to exist.

(a) The function attempts to approach multiple values as $x\to c$.
Geometrically, this behavior can be seen as a jump in the graph of a function.
Algebraically, this behavior typically arises with piecewise defined funtions.

(b) The function grows without bound as $x\to c$.
Geometrically, this behavior can be seen as a vertical asymptote in the graph of a function.
Algebraically, this behavior typically arises when the denominator of a function approaches zero.

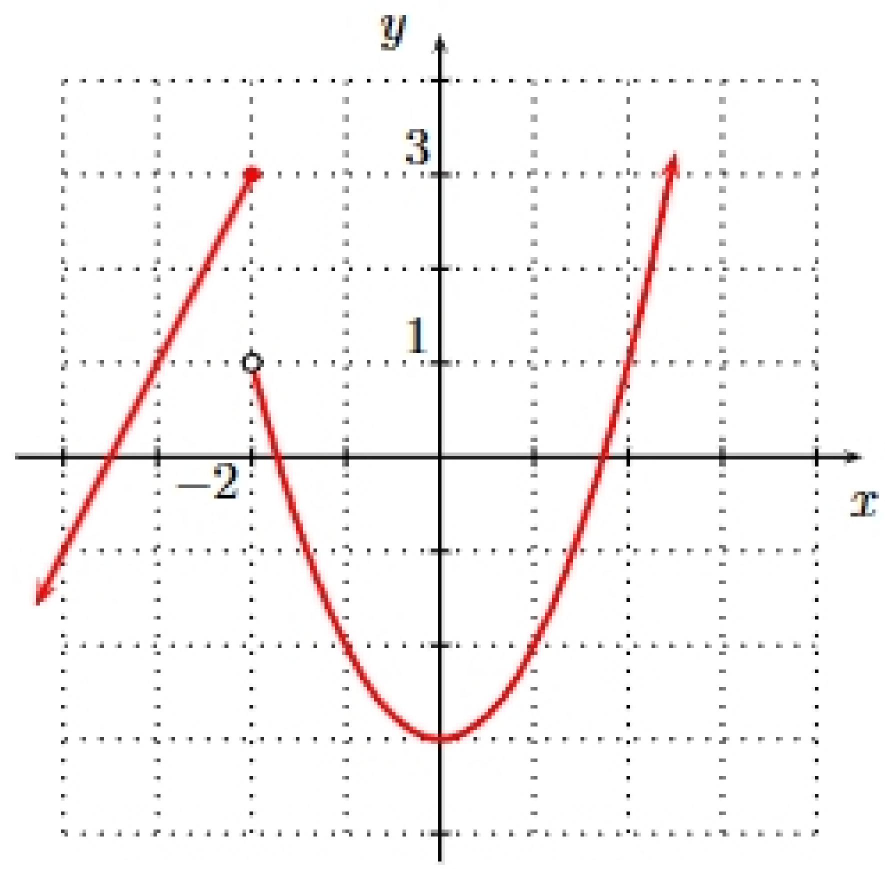
Example 5:

The graph of the function

$$h(x) = \begin{cases} x^2 - 3, & \text{if } x > -2; \\ 2x + 7, & \text{if } x \leq -2 \end{cases}$$

is shown to the right.

Analyze $\lim\limits_{x \to -2} h(x)$.



The previous example showed that the limit of a $h(x)$ as the variable approached -2 did not exist. On the other hand, the function appears to have well defined limiting behavior on either side of $x = -2$. This brings us to the following notions:

One-sided limits: A one-sided limit expresses what happens to the values of an expression as the variable in the expression gets closer and closer to some particular value c from either the left on the number line (that is, through values less than c) or from the right on the number line (that is, through values greater than c). The notation is:

$$\underbrace{\lim_{x \to c^-} f(x)}_{\text{limit from the left of } c} \qquad \underbrace{\lim_{x \to c^+} f(x)}_{\text{limit from the right of } c}$$

Fact: $\lim\limits_{x \to c} f(x)$ exists if and only if both $\lim\limits_{x \to c^-} f(x)$ and $\lim\limits_{x \to c^+} f(x)$ exist and have the same value.

Example 6:

The graph of the function

$$g(x) = \begin{cases} \dfrac{4x}{x^2 + 1}, & \text{if } x \neq 1; \\ 3, & \text{if } x = 1. \end{cases}$$

is shown to the right.

Compute $\lim\limits_{x \to 1} g(x)$.

x	$g(x)$
0.8	1.9512195
0.9	1.9889503
0.999	1.999999
1.001	1.999999
1.1	1.9909502
1.2	1.9672131