Mathematics 21b
Spring 2004

The subject: This is a course on *linear algebra and differential equations* with a special section that also introduces statistical techniques as used in the sciences. What follows is a brief description of the subject, plus a description of how the course is set up, course policies, and similar mundane matters.

As everyone in the course will be learning linear algebra and learning about differential equations, an introductory discussion is in order to explain what these subjects are about and why they should be understood by practicing scientists.

This introductory discussion has three parts, with the first two describing the roles of linear algebra and differential equations in the sciences. The third part says some things about the role of statistics.

Part 1: The discussion starts with the following definition of science:

The purpose of science is to predict future behavior from present circumstance.

Here is a somewhat simplistic elaboration:

The goal of a scientific program is to predict the outcome of experiments done in the future from some prescribed amount of presently known data.

For example, Watson and Crick's original proposal for genetic inheritance asserts that knowledge of the sequence of bases (adenine, guanine, cytosine, or thymine) along the DNA in a living cell is sufficient to predict the sorts of proteins that the cell can produce. Watson and Crick made this proposal based on their experimental understanding of the workings of a cell, and then further experimental work subsequently verified that their proposal is fundamental for understanding how cells encode their intrinsic operating instructions.

By the way, note how theory and experiment work hand in hand: Experiment tells us the present state of the world, theory provides the prediction for the future, and future experiments either falsify the theory or are consistent with its predictions. In this regard, a scientific theory must be falsifiable.

To a first approximation, a scientific theory can be viewed in the following abstract manner: The known data constitutes a labeled collection of numbers, $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$; here and below, integer subscripts play the role of labels. Meanwhile, data that arises from future experiments can always be labeled so as to constitutes a second ordered set of numbers, $\{y_1, \ldots, y_N\}$. Of course, the latter are not known until the experiments are carried out.

Granted this notation, a theory in science must give a well defined and reproducible method for predicting the future data, $\{y_1, \ldots, y_N\}$, from the initial data, $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. Thus, a theory can be viewed as a *function* that assigns an ordered collection of N numbers (the y's) to the collection of n numbers (the $\mathbf{x}$'s).

Of course, these experiments, once performed, provide a labeled set of N real data values, $\{y_1^{real}, \cdots, y_N^{real}\}$; and if each y_k^{real} is close to its predicted value, y_k, then the theory can be said to be an accurate description of reality. Of course, if some y_k^{real} is far from its prediction, y_k, the theory needs some revising.

The simplest non-constant functions are the *linear functions*; these have the schematic form

$$
\begin{aligned}
y_1 &= a_{11}\,\mathbf{x}_1 + a_{12}\,\mathbf{x}_2 + \cdots a_{1n}\,\mathbf{x}_n \\
&\cdot \\
&\cdot \\
&\cdot \\
y_N &= a_{N1}\,\mathbf{x}_1 + a_{N2}\,\mathbf{x}_2 + \cdots a_{Nn}\,\mathbf{x}_n
\end{aligned}
$$

where the collection $\{a_{ij}\}_{1\le i\le N, 1\le j\le n}$ are numbers. Thus, a scientific theory that is based on such a linear function would have to specify the collection $\{a_{ij}\}_{1\le i\le N, 1\le j\le n}$ and then knowledge of the input data $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ predicts the output, $\{y_1, \ldots, y_N\}$, of the future experiments using the preceding equation.

For an example, here is how this abstract view of a scientific theory can be implemented for the Watson-Crick theory of DNA coding: Each $\mathbf{x}_j$ can have an integer value between 1 and 4 so as to correspond to some chosen labeling of the four nucleotide basis. The subscript label $j \in \{1, \ldots, n\}$ designates the position of the given base along one strand of the stretch of DNA molecule under study. Meanwhile, each y_j has a value between 1 and 20 so as to correspond to some chosen labeling of one of the twenty biologically active amino acids. The subscript label $j \in \{1, \ldots, N\}$ for y_j signifies the position of the amino acid from some chosen end of the protein molecule.

As it turns out, the linear functions are among the most relevant to the sciences. This is because an appropriate linear function usually provides the mathematical equivalent of a 'first approximation' for a predictive description of any given phenomena. In any event, a good grasp of the mathematics of linear functions is a prerequisite for further explorations because the techniques that are used to study more complicated functions employ most of the mathematics for linear functions.

The subject of *Linear algebra* concerns the mathematics of linear functions.

Part 2: It is often the case that the quantity of interest in an experiment can be viewed as a function of some auxiliary variable. Indeed, consider the case when the quantity of interest changes with time, and so can be viewed as a function of time. Think of time as a variable, t, that can take values on the real line, and then the quantity of interest at time t can be written as a function of t. This is to say that there is a function of one variable, $t \to u(t)$, and the values of this function at time t are defined to be those of the quantity of interest. Here is a hypothetical example: Radioactive iodine is inadvertently dumped in a reservoir at time t = today. The concentration of iodine in the reservoir at any given future time can be called u(t), and so the assignment $t \to u(t)$ defines a function of time.

In any case, a full theoretical understanding of the time varying behavior of what is represented abstractly by one or several functions of time entails predicting their values at future times from present data. In the reservoir example, the present data might consist of the concentration of iodine measured today and, in addition, the average rates of inflow, outflow and evaporation of water from the reservoir.

As it turns out, theoretical predictions for the future behavior of real world quantities are often expressed via a system of equations that relate the *rate of change* of the quantities of interest at any given time to the values of the quantities at that same time. For example, such a theory for predicting the time dependence of some quantity that is modeled by a function $t \to u(t)$ would have the form

$$\frac{du}{dt} = f(u) ,$$

where the function f is specified by the theory. Such an equation is a simple example of a differential equation. In general, a differential equation can be said to be any equation for a function or set of functions that constrains the functions and their derivatives in some specified manner.

The subject of *differential equations* concerns techniques for finding solutions to differential equations. The subject also concerns techniques for estimating properties of interest of hypothetical solutions without knowledge of their explicit form.

Linear algebra enters the differential equation story in the following way: Just as linear functions are good first approximations for modeling many phenomenon, so differential equations that involve linear functions also offer reasonable first approximations in many situations. As is explained in this course, differential equations that involve only the linear functions can almost always be solved by closed form expressions using linear algebra. In this regard, note that most of the known techniques for dealing with non-linear differential equations involve generalizations of those that work for the class of linear differential equations.