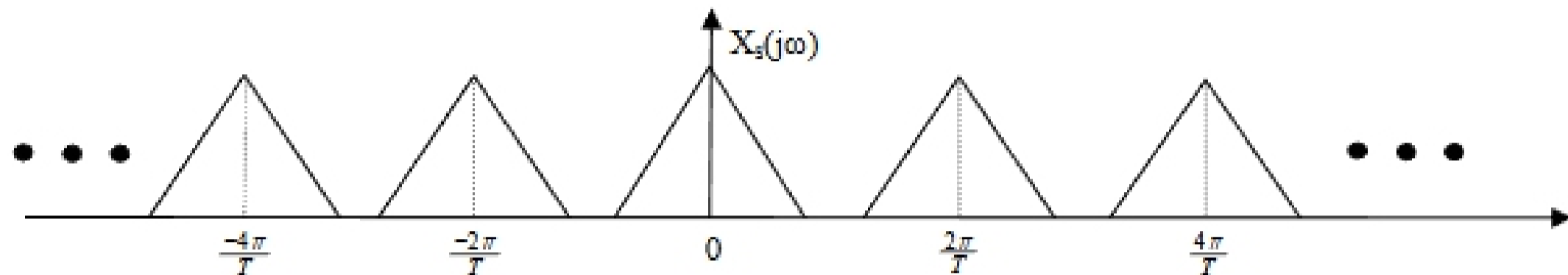


8-4B Steady-State Frequency Response of a Linear Discrete-Time System

In this section, we study various properties of the discrete-time Fourier Transform.

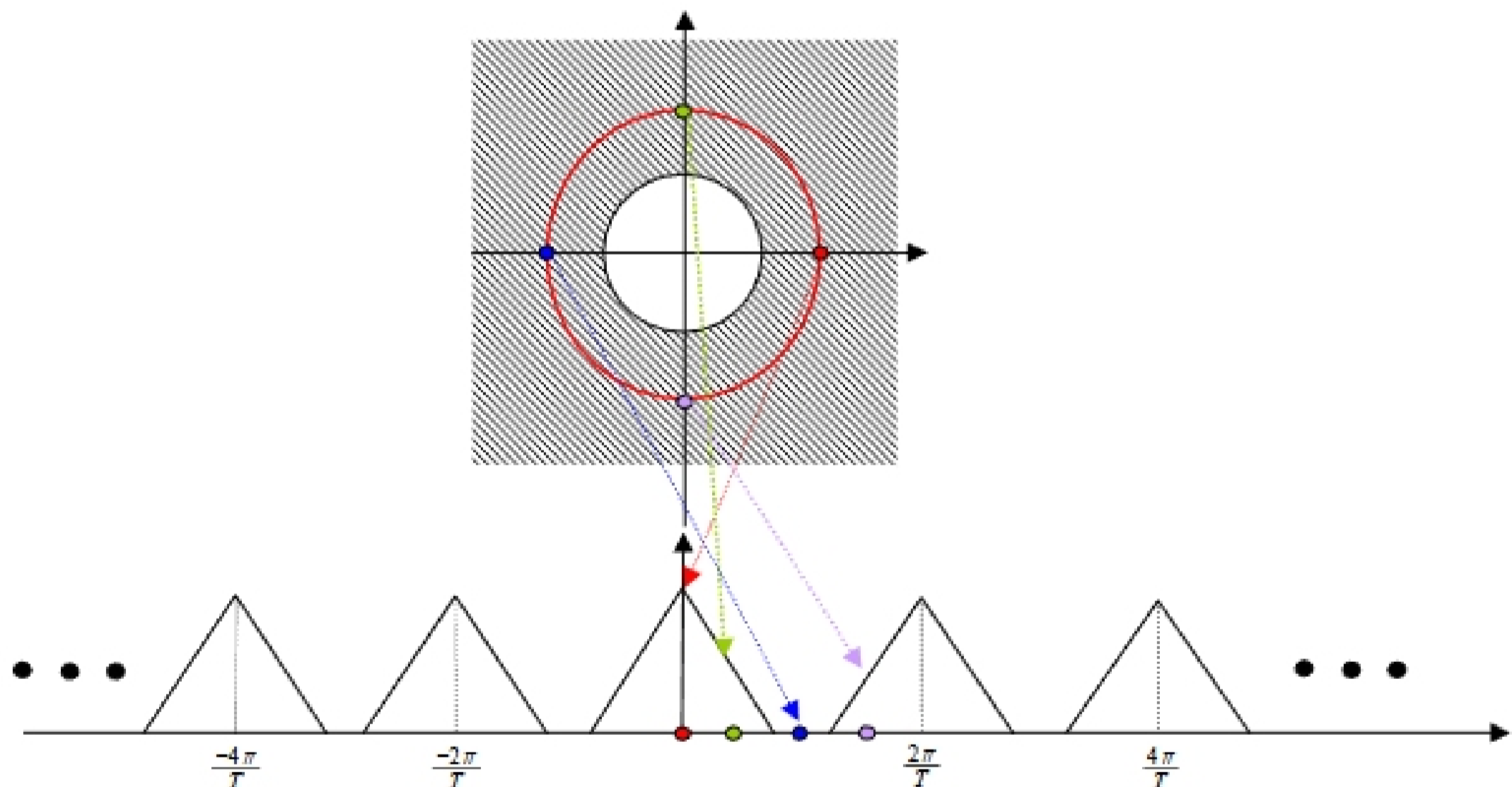
As we stated earlier, it is the same as the continuous-time Fourier Transform of the sampled signal $x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$:



We have also claimed that it can be computed by evaluating the Z-transform around the unit circle (provided that the unit circle is inside the ROC)

Definition of DTFT: $X(e^{j\omega T}) = X(z)|_{z=e^{j\omega T}} = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$

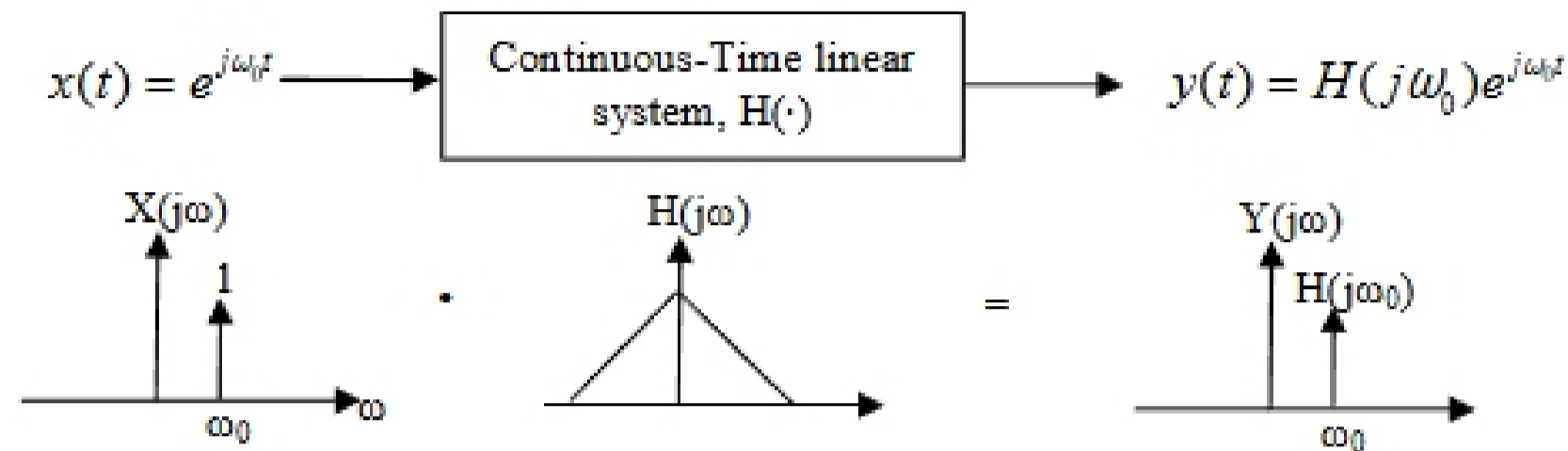
Note that we use $X(e^{j\omega T})$ to indicate the substitution $z=e^{j\omega T}$.



Let's first show that these two definitions are equivalent. Start with the continuous-time Fourier Transform of the sampled signal:

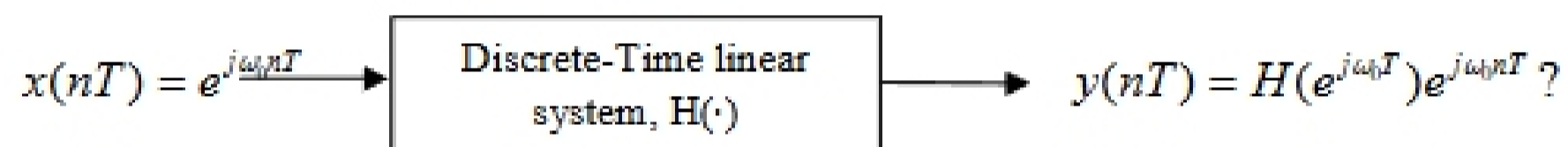
$$X_s(j\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)e^{-j\omega t} dt = \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} e^{-j\omega t} \delta(t - nT) dt = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT} = X(e^{j\omega T})$$

The primary reason to study Fourier Transform in continuous-time linear system is that *if the input to a continuous-time linear system is a complex sinusoid of frequency ω_0 , the output is also a complex sinusoid of frequency ω_0 with a phase shift and a gain governed by $H(j\omega_0)$.*



This form of analysis, as you know, is called the steady state analysis (steady state as complex sinusoid is not transient).

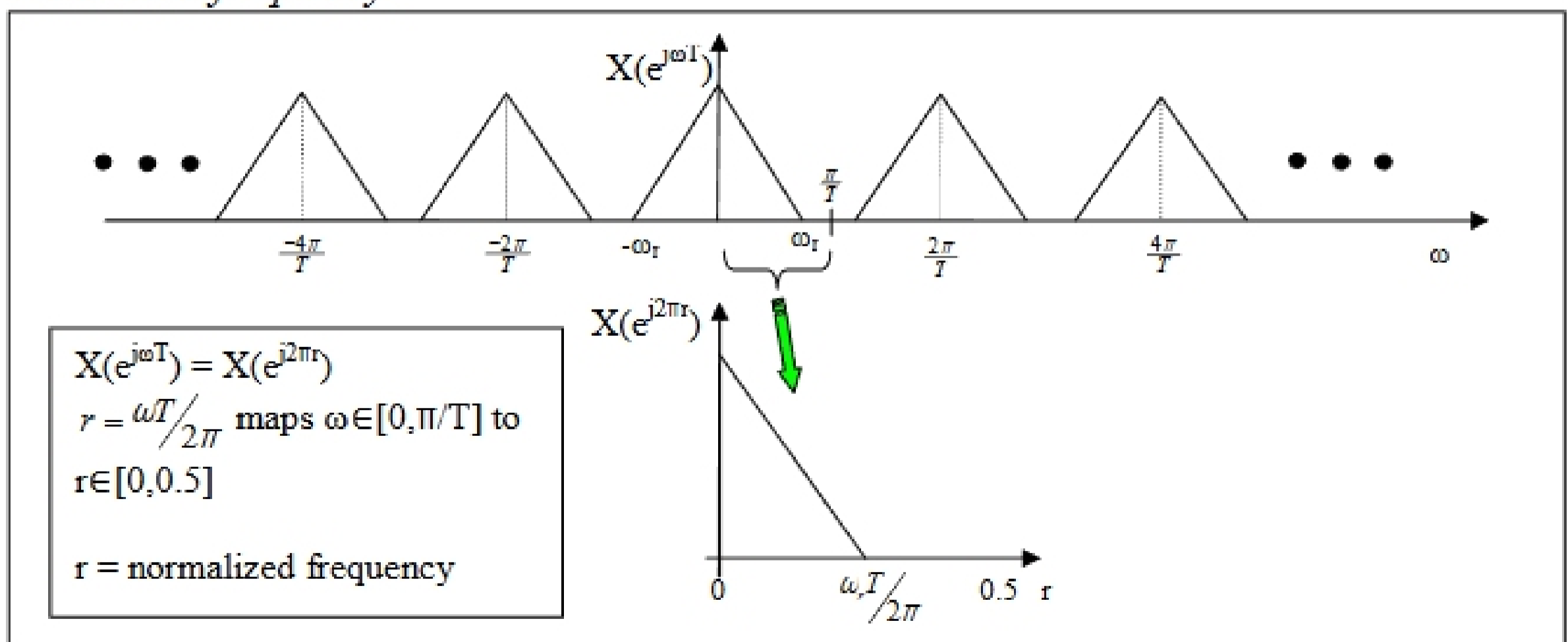
It would be nice if the DTFT can do the same for Discrete-time linear system or



And indeed it is true:

$$\begin{aligned} y(nT) &= \sum_{m=0}^{\infty} x(nT - mT)h(mT) \\ &= \sum_{m=0}^{\infty} e^{j\omega_0(n-m)T}h(mT) \\ &= e^{j\omega_0 nT} \sum_{m=0}^{\infty} e^{-j\omega_0 mT}h(mT) \\ &= e^{j\omega_0 nT} H(e^{j\omega_0 T}) \end{aligned}$$

Before we go on, let's introduce a common representation of the DTFT based on *normalized frequency*.



Remarks

1. Why called normalized frequency?

Given the sampling frequency $f_s = 1/T$ and the frequency $f = \omega/2\pi$, r can be more succinctly represented as

$$r = \omega T / 2\pi = f / f_s$$

When using normalized frequency r , the DTFT is written as $H(e^{j2\pi r})$.

2. Why ignore the negative frequency?

For real-valued $h(nT)$, we can deduce the negative frequency from the positive frequency:

$$\begin{aligned} H(e^{j(-\omega)T}) &= \sum_{n=-\infty}^{\infty} h(nT)e^{j\omega nT} \\ &= \left[\sum_{n=-\infty}^{\infty} h(nT)e^{-j\omega nT} \right]^* \quad (* = \text{conjugate}) \\ &= H^*(e^{j\omega T}) \Rightarrow |H(e^{-j\omega T})| = |H(e^{j\omega T})| \ \& \ \angle H(e^{-j\omega T}) = -\angle H(e^{j\omega T}) \end{aligned}$$

Thus, it is sufficient to show only the positive part only.

The followings are a list of DTFT properties and common transform pairs. They can be easily deduced from the Z-transform tables.

TABLE 2.2 FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$

Parseval's theorem:

$$8. \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

$$9. \sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$