

V.3 Piecewise Linear Functions

In practical situations we rarely (or perhaps never) have smooth functions. Instead, we use other functions to approximate smooth functions. In this section, we use insights gained into the smooth case as a guide in our attempt to understand the piecewise linear case.

Lower star filtration. Let K be a simplicial complex with real values specified at all vertices. Using linear extension over the simplices, we obtain a *piecewise linear (PL) function* $f : K \rightarrow \mathbb{R}$. It is defined by $f(x) = \sum_i b_i(x)f(u_i)$, where the u_i are the vertices of K and the $b_i(x)$ are the barycentric coordinates of x ; see Section III.1. To emphasize that f is linear on every simplex, we prefer the above notation over the more accurate $f : |K| \rightarrow \mathbb{R}$. It is convenient to assume that f is *generic* by which we mean that the vertices have distinct function values. We can then order the vertices by increasing function value as $f(u_1) < f(u_2) < \dots < f(u_n)$. For each $0 \leq i \leq n$, we let K_i be the full subcomplex defined by the first i vertices. In other words, a simplex $\sigma \in K$ belongs to K_i iff each vertex u_j of σ satisfies $j \leq i$. Recall that the star of a vertex u_i is the set of cofaces of u_i in K . The *lower star* is the subset of simplices for which u_i is the vertex with maximum function value,

$$\text{St}_- u_i = \{\sigma \in \text{St } u_i \mid x \in \sigma \Rightarrow f(x) \leq f(u_i)\}.$$

By assumption of genericity, each simplex has a unique maximum vertex and thus belongs to a unique lower star. It follows that the lower stars partition K . Furthermore, K_i is the union of the first i lower stars. This motivates us to call the nested sequence of complexes $\emptyset = K_0 \subset K_1 \subset \dots \subset K_n = K$ the *lower star filtration* of f . It will be useful to notice that the K_i are representative of the continuous family of sublevel sets. Specifically, for $f(u_i) \leq t < f(u_{i+1})$

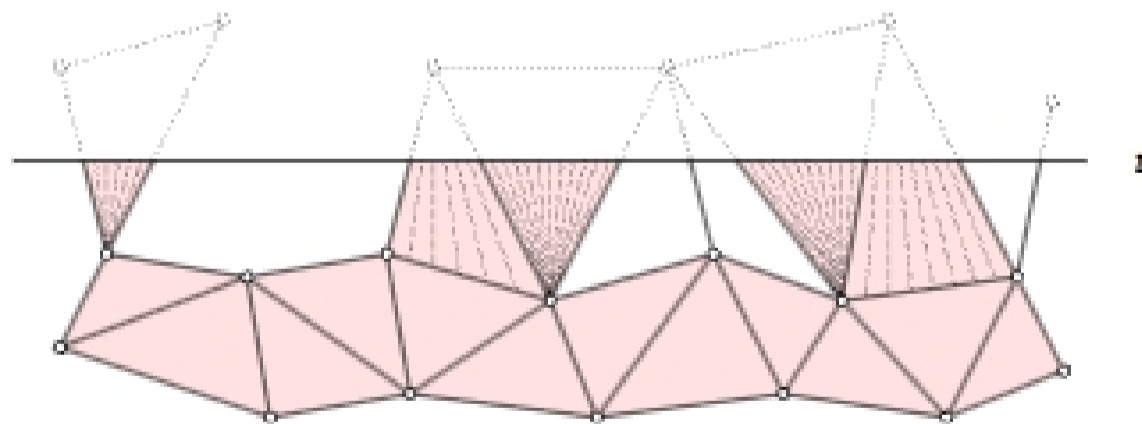


Figure V.8: We retract $|K|_t$ to K_i by shrinking the line segments decomposing the partial simplices from the top downward.

the sublevel set $|K|_t = f^{-1}(-\infty, t]$ has the same homotopy type as K_i . To prove this consider each simplex with at least one vertex in K_i and at least one in $K - K_i$. Write this simplex as a union of line segments connecting points on the maximal face in K_i with points on the maximal face in $K - K_i$, as shown in Figure V.8. The sublevel set $|K|_t$ contains only a fraction of each such line segment, namely the portion from the lower endpoint x in $|K_i|$ to the upper endpoint y with $f(y) = t$. To get a deformation retraction we let $y_\lambda = \lambda x + (1 - \lambda)y$ be the upper endpoint at time λ . Going from time $\lambda = 0$ to $\lambda = 1$ proves that the sublevel set for t and K_i have the same homotopy type.

PL critical points. Let us study the change from one complex to the next in the lower star filtration in more detail. Recall that the link of a vertex is the set of simplices in the closed star that do not belong to the star. Similarly, the *lower link* is the collection of simplices in the lower star that do not belong to the lower star. Equivalently, it is the collection of simplices in the link whose vertices have smaller function value than u_i ,

$$\text{Lk}_- u_i = \{\sigma \in \text{Lk } u_i \mid x \in \sigma \Rightarrow f(x) < f(u_i)\}.$$

When we go from K_{i-1} to K_i we attach the closed lower star of u_i , gluing it along the lower link to the complex K_{i-1} . Assume now that K triangulates a d -manifold. This restricts the possibilities dramatically since every vertex star is an open d -ball and every vertex link is a $(d - 1)$ -sphere. A few examples of lower stars and lower links in a 2-manifold are shown in Figure V.9. We classify

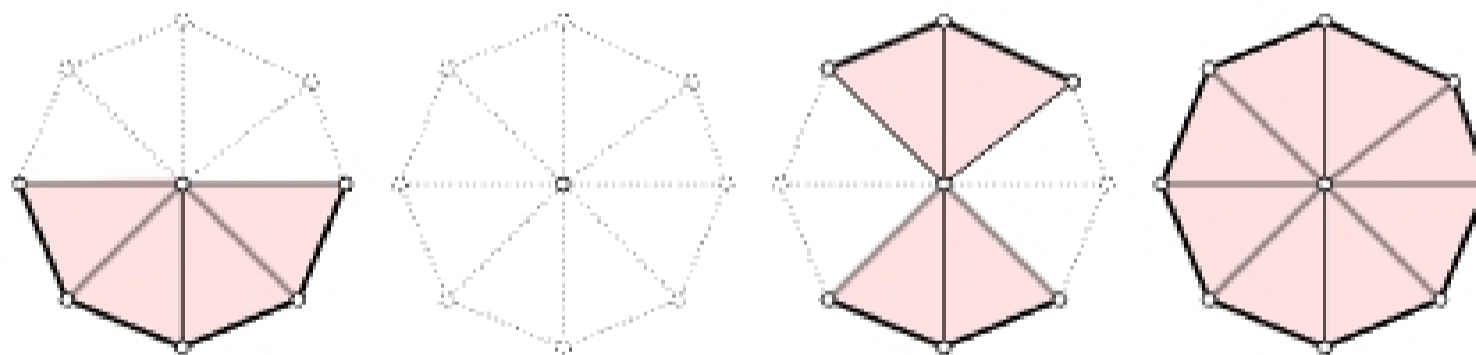


Figure V.9: From left to right: the lower star and lower link of a regular vertex, a minimum, a saddle, and a maximum.

the vertices using the reduced Betti numbers of their lower links. Recall that $\tilde{\beta}_0$ is one less than β_0 , the number of components. The only exception to this rule is the empty lower link for which we have $\tilde{\beta}_0 = \beta_0 = 0$ and $\tilde{\beta}_{-1} = 1$. Table V.1 gives the reduced Betti numbers of the lower links in Figure V.9. We call u_i a *PL regular vertex* if its lower link is non-empty but homologically trivial and we call u_i a *simple PL critical vertex of index p* if its lower link has

	$\bar{\beta}_{-1}$	$\bar{\beta}_0$	$\bar{\beta}_1$
regular	0	0	0
minimum	1	0	0
saddle	0	1	0
maximum	0	0	1

Table V.1: Classification of the vertices in a PL function on a 2-manifold.

the reduced homology of the $(p-1)$ -sphere. In other words, the only non-zero reduced Betti number of a simple PL critical vertex of index p is $\bar{\beta}_{p-1} = 1$.

DEFINITION. A piecewise linear function $f : K \rightarrow \mathbb{R}$ on a manifold is a *PL Morse function* if (i) each vertex is either PL regular or simple PL critical and (ii) the function values of the vertices are distinct.

Unfolding. In contrast to the smooth case, PL Morse functions are not dense among the class of all PL functions. Equivalently, a PL function on a manifold may require a substantial perturbation before it becomes PL Morse. As an example consider the piecewise linear version of a monkey saddle displayed in Figure V.10. It is therefore not reasonable to assume a PL Morse function

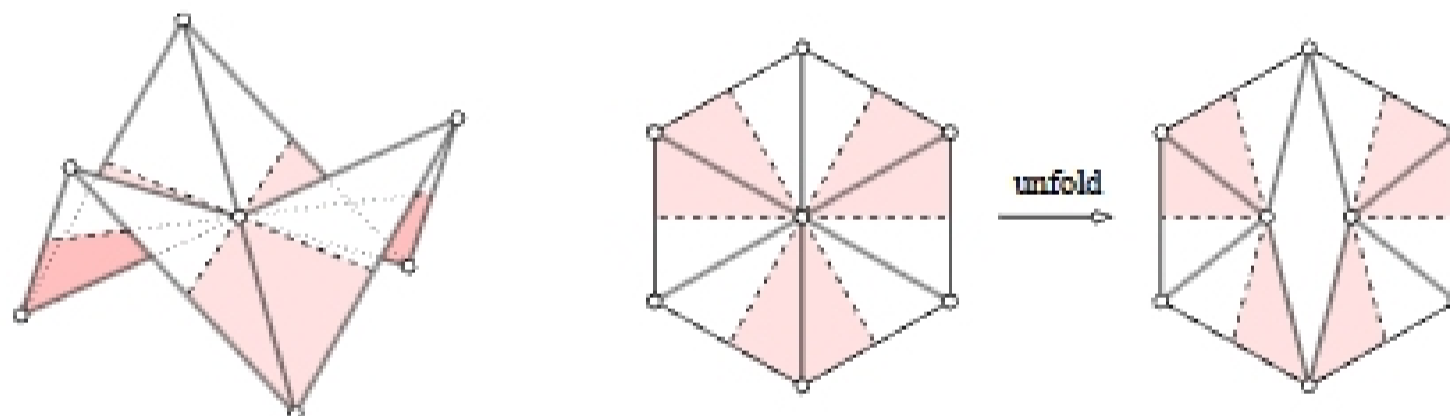


Figure V.10: Left: a PL monkey saddle of a height function. The areas of points lower than the center vertex are shaded. Right: the unfolding of the PL monkey saddle into two simple saddles.

as input, but we can sometimes alter the triangulation locally to make it into a PL Morse function. In the 2-manifold case, a k -fold saddle is defined by $\bar{\beta}_0 = k$. We can split it into k simple saddles by introducing $k-1$ new vertices and assigning appropriate function values close to that of the original, k -fold saddle; see Figure V.10 for the case $k=2$. It is less clear how to unfold possibly complicated PL critical points for higher-dimensional manifolds; see Section IX.8.