# ME 406
# The Lorenz Equations

```
sysid
```

```
Mathematica 4.1.5, DynPac 10.67, 4/8/2002
```

```
intreset;
```

```
plotreset;
```

## 1. Introduction

This notebook contains all of the material given in class on the Lorenz equations, and it constitutes section 2.5 of the class notes. The Lorenz equations are given by

$$\dot{x} = \sigma(y - x)\,,\ \ \dot{y} = rx - y - xz\,,\ \ \dot{z} = xy - bz\ . \qquad (1)$$

These equations contain three parameters: $\sigma$, $r$ and $b$. In what follows, we will always assume that these parameters are positive. In all of the numerical calculations below, we take $\sigma = 10.0$, and $b = 8/3$. These are the most used values in the study of the Lorenz equations. We will vary the parameter $r$ over a wide range, and study how the solutions depend on $r$.

These equations, which are simple in appearance, have solutions with extraordinary properties. They were first studied in the 1960's by the M.I.T. meteorologist Edward Lorenz. He developed the equations as a model for the modal amplitudes in a nonlinear thermal convection problem. Lorenz recognized that the solutions of the equations can exhibit an unusual form of behavior which we now call chaos. It took time for others to realize exactly what Lorenz had discovered. Lorenz has told the story of the discovery in his book **The Essence of Chaos,** University of Washington Press, 1993. For a very readable and basic treatment of the equations, see Chapter 9 of **Nonlinear Dynamics and Chaos**, S.H. Strogatz, Addison-Wesley, 1994. For a general work containing a more advanced treatment, see **Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields**, J. Guckenheimer and P. Holmes, Springer-Verlag, 1983. For a book-length treatment containing many detailed results, see **The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors**, C. Sparrow, Springer-Verlag, 1982. In our study here, we will continue our primarily experimental approach and use the computer to learn about the system. We begin by defining the equations for DynPac.

```
setstate[{x, y, z}]; setparm[{σ, r, b}]; sysname = "Lorenz";
```

```
slopevec = {σ * (y - x), r * x - y - x * z, x * y - b * z};
```

## 2. Basic Properties

### 2.1 Symmetry

The equations are invariant to the transformation

$$(x, y, z) \longrightarrow (-x, -y, z) \ . \tag{2}$$

Thus if $x(t)$, $y(t)$, $z(t)$ is a solution, so is $-x(t)$, $-y(t)$, $z(t)$. As we shall see, this symmetry shows up in a number of ways, including the location of the equilibrium points of the system.

### 2.2 The z-Axis is Invariant

We see from equations (1) that if $x(0) = 0$ and $y(0) = 0$, then $x$ and $y$ remain zero for all $t$. Thus the $z$-axis is an orbit, on which

$$\dot{z} = -bz \text{ , hence } z(t) = z(0)e^{-bt} \text{ , for } x, y = 0. \tag{3}$$

Thus the $z$-axis is always a part of the stable manifold for the equilibrium at the origin.

### 2.3 The System is Dissipative

The divergence of the slopevector is

```
div[slopevec]
```

```
-1 - b - σ
```

and this is always negative. As we saw earlier in class, for any given volume $V$ of phase points moving with the flow, we have

$$\frac{dV}{dt} = V\,\text{div[slopevec]} = -(1 + b + \sigma)V, \qquad \text{hence } V(t) = V(0)e^{-(1+b+\sigma)\,t} \ . \tag{4}$$

With our canonical values of 10 for $\sigma$ and 8/3 for $b$, this is $V(t) = V(0)e^{-13.67\,t}$, so that volumes of initial points are reduced by a factor of $e$ in a time

```
1 / 13.67
```

```
0.0731529
```

### 2.4 The Solutions are Bounded

It is not hard to prove that the solutions of the Lorenz equations are bounded. Consider first the case when $r < 1$. We examine a potential Liapunov function

```
V = x^2 + σ * (y^2 + z^2);
```

The orbital derivative is

```
Simplify[orbdt[V]]
```

```
-2 (x^2 - (1 + r) x y + y^2 + b z^2) σ
```

The $z$-term is negative. We get for the $x$ and $y$ terms

$$x^2 - (1 + r)\,x\,y + y^2 = \left[x - \frac{1}{2}\,(1 + r)\,y\right]^2 + \frac{1}{4}(1 - r)(3 + r)\,y^2.$$

and this quantity is postive except for $x = y = 0$. Thus $\dot{V} < 0$, and this shows that the origin is a global attractor for the system when $r < 1$.

When $r > 1$, we have to work a little harder to show the solutions are bounded. We start with a new Liapunov function

```
V = r * x^2 + σ * y^2 + σ * (z - 2 * r)^2;
```

Level surfaces $V = V_0$ are ellipsoids with center at $(0, 0, 2r)$, and semi-axes $(\sqrt{V_0/r}, \sqrt{V_0/\sigma}, \sqrt{V_0/\sigma})$. The orbital derivative is

```
Vdot = Simplify[orbdt[V]]
```

$$-2\,(y^2 + b\,z^2 + r\,(x^2 - 2\,b\,z))\,\sigma$$

We work on this a bit.

```
Vdotmod = Vdot / (2 σ b r^2)
```

$$-\frac{y^2 + b\,z^2 + r\,(x^2 - 2\,b\,z)}{b\,r^2}$$

We can write this as

$$-\frac{x^2}{br} - \frac{y^2}{b\,r^2} - \frac{(z-r)^2}{r^2} + 1 \qquad (5)$$

The expression in equation (5) is zero on the ellipsoid given by

$$\frac{x^2}{br} + \frac{y^2}{b\,r^2} + \frac{(z-r)^2}{r^2} = 1 \ , \qquad (6)$$

and is positive inside that ellipsoid and negative outside that ellipsoid. By choosing $V_0$ large enough, we can make the level surface $V = V_0$ lie entirely outside the ellipsoid (6). Any orbit starting outside that level surface will cross to the inside, and any orbit starting inside that level surface will remain inside. Thus all of the solutions are bounded, subject only to the assumption we made at the beginning that $\sigma$, $r$, and $b$ are positive. Unlike the case $r < 1$, we cannot in this case draw any conclusions about a global attractor or even a stable equilibrium.

## ▪ 2.5 Equilibria

We use findpolyeq to look for equilibria.

```
eqpoints = findpolyeq
```

$$\{\{0, 0, 0\}, \{-\sqrt{-b + b\,r}, -\sqrt{-b + b\,r}, -1 + r\}, \{\sqrt{-b + b\,r}, \sqrt{-b + b\,r}, -1 + r\}\}$$

We see that the origin is an equilibrium for any values of the parameters. The other two equilibria are real if and only if $r \geq 1$. In fact for $r = 1$, the three equilibria coincide at {0,0,0}, and we therefore have a pitchfork bifurcation at $r = 1$. We will look at this in more detail later. We name the three equilibria for later reference.