



Energy in magnetic fields

In electrostatics, we found that the potential energy of an arrangement of charges, and the potentials and fields they create, was given by

$$W = \frac{1}{2} \int_V \rho V dt \quad \text{in general, and}$$

$$= \frac{1}{8\pi} \int_{\text{space}} E^2 dt$$

if the product of E , V , and r^2 approach zero as $r \rightarrow \infty$ (see lecture notes for 25 September 2002). What have we for magnetism that corresponds to this?

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Energy in magnetic fields (continued)

To find such an expression, let's exploit $W = LI^2/2$, and consider a surface S , threaded by magnetic field and bounded by curve C

$$\Phi_B = \int_S B \cdot da = \int_S (\nabla \times A) \cdot da = \oint_C A \cdot dt$$

$$= cLI \quad , \quad \text{so} \quad LI = \frac{1}{c} \oint_C A \cdot dt \quad .$$

Thus $W = \frac{1}{2} LI^2 = \frac{I}{2c} \oint_C A \cdot dt = \frac{1}{2c} \oint_C (A \cdot I) dt \quad .$

But $I = \oint_{S'} J da$, so $W = \frac{1}{2c} \int_V (J \cdot A) dt \quad . \quad \left[\text{cf. } W = \frac{1}{2} \int_V \rho V dt \right]$

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We can use Ampère's law to eliminate J from this expression,

$$W = \frac{1}{2c} \int_V \left(\frac{c}{4\pi} \nabla \times \mathbf{B} \right) \cdot \mathbf{A} d\tau \quad ,$$

and simplify the result using Product Rule #6:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \quad ,$$

$$\mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mathbf{B}^2 - \nabla \cdot (\mathbf{A} \times \mathbf{B}) \quad ,$$

to get

$$W = \frac{1}{8\pi} \int_V \left(\mathbf{B}^2 - \nabla \cdot (\mathbf{A} \times \mathbf{B}) \right) d\tau$$

$$= \frac{1}{8\pi} \int_V \mathbf{B}^2 d\tau - \frac{1}{8\pi} \oint_{\mathcal{S}'} (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a} \quad .$$

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where we have used the divergence theorem, and identified the surface \mathcal{S}' that bounds V . Now suppose we extend the volume V to fill all space -- which we can do without penalty, if no additional currents are encompassed thereby. Then the surface integral (over \mathcal{S}') vanishes, since \mathbf{A} and \mathbf{B} approach zero very far from the currents.

□ This is a cleaner case than that of electrostatics. Remember that this step runs into trouble with point electric charges (lecture notes, 25 September 2002), for which the surface integral doesn't vanish. There aren't any point magnetic charges. In fact, there aren't any magnetic charges (monopoles) of any extent.

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Energy in magnetic fields (continued)

So

$$W = \frac{1}{8\pi} \int_{\text{all space}} \mathbf{B}^2 d\tau \quad , \quad \left[\text{cf. } W = \frac{1}{8\pi} \int_{\text{all space}} \mathbf{E}^2 d\tau \right]$$

If there's both \mathbf{E} and \mathbf{B} present, we thus get the neat result

$$W = \frac{1}{8\pi} \int_{\text{all space}} \left(\mathbf{E}^2 + \mathbf{B}^2 \right) d\tau \quad .$$

$$\left[W = \frac{1}{2} \int_{\text{all space}} \left(\epsilon_0 \mathbf{E}^2 + \frac{\mathbf{B}^2}{\mu_0} \right) d\tau \text{ in MKS units.} \right]$$

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