

Lesson 28: A Further Look at Inverse Functions

(Cover 8.2)

Read: Section 8.3

Do: WebWork, Team Homework

The most important points and skills for §8.2

- Students are able to describe the meaning of the term *invertible function* using something similar to: “A function is an invertible function if “the inverse” is also a function in its own right.”
- Given a function (defined by a table, an explicit formula, or a graph) students are able to use the horizontal line test to determine whether or not the inverse will also be a function in its own right.
- Given an invertible function defined by an explicit formula, students are able to find an explicit formula for the inverse.
- Given a function defined by a graph, students are able to sketch the graph of the inverse.
- Given a function defined by a table or a graph, students are able to approximate values of the inverse. For example, if given a graph and asked to find the value of $f^{-1}(10)$, students know to find the x -coordinate of the point that corresponds to a height of 10 on the graph.
- Given a “real world” context and a function that relates two quantities in the context, students are able to give “real world” interpretations of the meaning of statements written in function notation or inverse function notation.
- Given a function that is not invertible, students are able to restrict the domain in order to find inverse functions (e.g. inverse trigonometric functions).

Comment: In high school a lot of students will have learned a procedure along the lines of the following for finding a formula for the inverse of a function:

1. Start with the formula of the function.
2. Swap the x 's and the y 's in this formula.
3. Manipulate to make y the subject, i.e. “solve for y in the new formula”.

The book, on the other hand, omits the “swap x and y ” step (see pages 363-364). The advantage of swapping is that students are more accustomed to solving for y and so are less likely to become confused as to what algebraic manipulations are needed in Step 3. The (huge) disadvantage is that if the x and the y actually stand for real-world quantities then you also have to swap the meaning of x and y when you calculate the formula for the inverse. This is a tough point for a lot of students. Namely, if x is in dollars and y is in apples in the original function, then they expect x to be in dollars and y to be in apples in the inverse as well. It is perhaps for this reason that the book advocates an approach to computing inverses in which you simply re-arrange an equation as in the following example.

Example. The population of a small city was 20,000 in the year 2005. The population, P , is a function of time, t , where P is measured in thousands of people and t is the number of years after 2005. Specifically, $P = f(t) = 20e^{0.05t}$. We can determine what year the population reaches a

certain value by writing t as a function of P . (The name of this new function happens to be f^{-1} .) Solving for t , we have

$$\begin{aligned}P &= 20e^{0.05t} \\ \frac{P}{20} &= e^{0.05t} \\ \ln\left(\frac{P}{20}\right) &= 0.05t \\ 20 \ln\left(\frac{P}{20}\right) &= t\end{aligned}$$

So we have now found the inverse function, $t = f^{-1}(P) = 20 \ln\left(\frac{P}{20}\right)$.

**The part that might get particularly confusing for students is what to do if they wanted to plot the inverse and the original function on the same set of axes. You can point out that once you find the inverse, you can use different variables as inputs, i.e. you could plot $f(x)$ and $f^{-1}(x)$ on the same xy -axes. **

Suggested Lesson Plan:

00–25 Use this time for a quiz or to do more examples from the last lesson. Some possibilities include **Section 8.1 #3, 26, 34, and 37 on pages 360–361.**

25–30 You could begin the new material with a very short and focused mini-lecture aimed at communicating the definition of an inverse function and an invertible function to the students, such as what is presented on page 362. Recall what the students learned back in Chapter 2 by means of notation and interpretation and tell them that now they will also be able to graph and find formulas for inverse functions.

Tables

30–40 You could then ask students to work on **Section 8.2 #40(a) and (b) on page 371.** Although students have worked with the concept of the inverse before, tables can still be quite confusing. Discuss how the definition of an inverse function and an invertible function can be used here to decide whether or not $f(x)$ and $g(x)$ could be invertible. Remember that some students had plenty of trouble deciding whether or not the values in a table represented a function. Expect them to have at least as much trouble deciding whether or not a table of values represents an *invertible function*. At the end of this discussion you could summarize the discussion with a short mini-lecture on how to tell whether or not a table of values represents (or could represent) an invertible function or not.

Graphs

40–55 Next, you could conduct an interactive mini-lecture on the horizontal line test. You could start by asking the students if they know how to tell whether or not a function defined by a graph is invertible. At least some of the students will be familiar with the horizontal line test (and they *all* should be based on their reading), but you should summarize what it says, how to use it, and why it works for the students who have not had it before. Ask the students to do **Section 8.2 #4 and #6 on page 370.**

When you have finished discussing the horizontal line test, you could remind students how to draw the graph of the inverse of a function by reflecting the graph of $y = f(x)$ over the line $y = x$. (See Example #7 on pages 366 and 367.) Put the graph of an invertible function on the board (something like $f(x) = x^3$ would work here) and ask students to sketch the inverse function.

Next show the sine function as an example of a function that does NOT pass the horizontal line test. Point out that we can restrict the domain so that the function over the restricted domain will pass the horizontal line test. Draw the connection between this idea and how we had restricted ranges for inverse trigonometric functions. Have the students sketch the graph of $y = \sin^{-1}(x)$ over the correct domain and range. As time allows, you could ask the students to draw the graphs of $y = \cos^{-1}(x)$ and $y = \tan^{-1}(x)$.

Formulas

55–70 You should then go through a few examples of calculating formulas for the inverses of invertible functions. **Section 8.2 #17 – 25 (odd) on page 370** are all suitable for use as examples here. Do one as an example, *making sure to follow the same method that is outlined in the book in Example 5 on page 364*. Then ask the students to work in their groups to complete additional problems. Students will require **lots** of practice with this since algebra tends to be a problem.

Interpretation

70–80 End the lesson by having the students work on problems in which they have to evaluate and interpret function and inverse notation in real-world context, such as a sensible selection among **Section 8.2 #52–57 on page 372**, given the time you have left. Make sure the students clearly note appropriate units for both the inputs as well as the outputs for the given expressions.

Summarize the key ideas of the lesson.