Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work. **Show all relevant supporting steps!**

1. Let $z = \sqrt{3} - i$ and $w = -2 + 2i$. Find $\frac{w^3}{z^9}$ and express the value in rectangular form.

2. Let (X,d) be a metric space and let $A \subset X$. Prove: *A is closed if and only if* $A = \overline{A}$.

3. Classify the following sets as to whether they are: a) open, b) closed, c) connected, d) polygonally path-connected, e) compact, f) complete, g) bounded, h) region.

   You do not need to provide a rationale for your classification. Use the table on the last page to record your classifications.

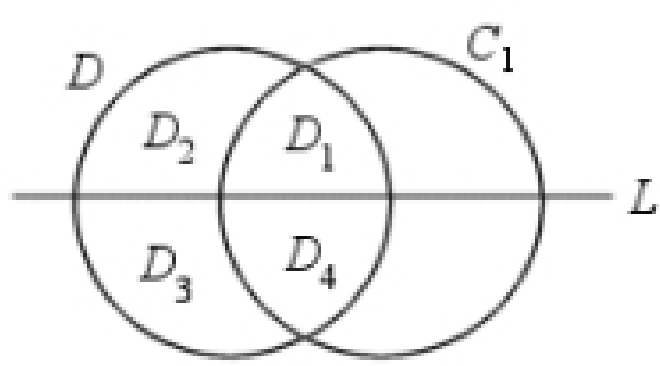
   1. $A = \{(x,y) : 0 < y < \frac{1}{x}|\sin\frac{1}{x}|,\ 0 < x < \pi\}$
   2. $B = B(1,1) \setminus \overline{B(1/2, 1/2)}$
   3. $C = S \setminus \overline{B(0,\pi)}$ where $S = \{z : |\mathrm{Re}\, z| < 1\}$
   4. $D = B(2,2) \setminus \overline{B(2-2i, \sqrt{8})}$

4. Find the radius of convergence of the power series

   1. $\sum_{n=0}^{\infty} \frac{2^n n! n!}{(2n)!}(3z+1)^n$
   2. $\sum_{n=0}^{\infty} a_n (z+i)^n$ where the power series represents $f(z) = \tan z$

5. Let $G = \{z = re^{i\theta} : 0 < r < 1,\ 0 < \theta < \pi/2\}$ and let $S = \{z : |\mathrm{Re}\, z| < 1\}$. Find a one-to-one conformal map from G to S.

6. The line $L = \{z : \mathrm{Im}\, z = 0\}$ and the circle $C_1 = \{z : |z-1| = 1\}$ divide $D = \{z : |z| < 1\}$ into 4 subregions D_1, D_2, D_3 and D_4. See figure to the right. Let $w = \frac{z}{z-2}$. Find the images E_j of each subregion D_j under w, i.e., find $E_j = w(D_j)$, $j = 1,2,3,4$.



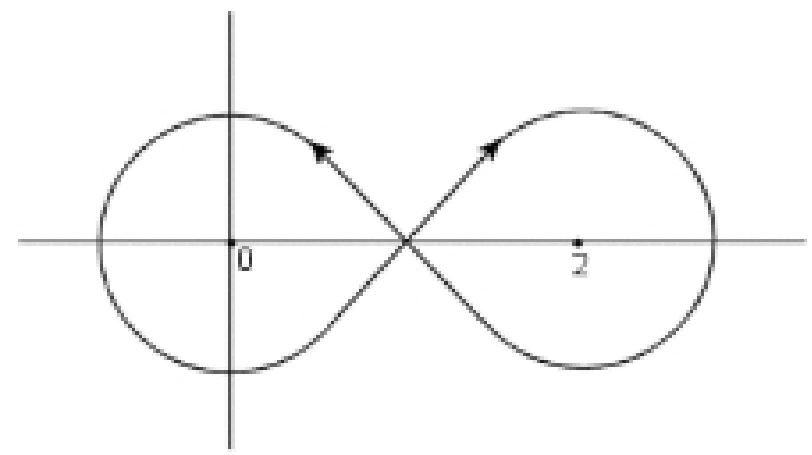
7. Let G be a region in $\mathbb{C}$ and let $f \in \mathcal{A}(G)$, $f = u + iv$. Prove that if $u^2 - v^2 = 1$ on G, then f is constant on G.

8. Evaluate each of the following integrals:

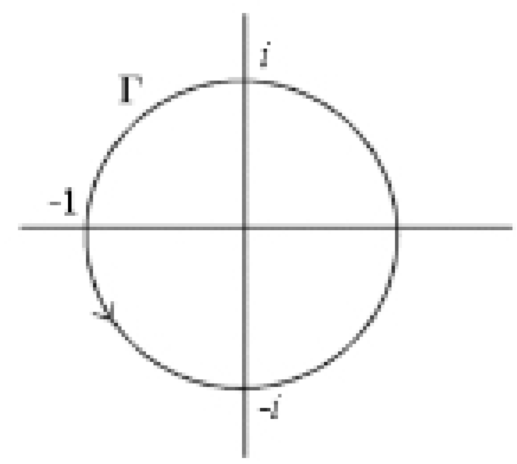
1. $\int_{\gamma} (\operatorname{Re} z + (\operatorname{Im} z)^2) dz$ where γ is a parametrization of the straight line segment from 1 to i.

2. $\int_{\gamma} \frac{\cos z + 1}{z^3} dz$ where γ is a positively (counter-clockwise) oriented parametrization of the unit circle $C = \{z : |z| = 1\}$

3. $\int_{\gamma} \frac{z+1}{z^2 - 2z} dz$ where γ is a parametrization of the figure eight curve oriented as given in the figure to the right



4. $\int_{\gamma} \frac{dz}{\sqrt{z}}$ where γ is a parametrization of the arc Γ of the unit circle $C = \{z : |z| = 1\}$ from i to $-i$ passing through -1 (see figure to the right) and where $\sqrt{z}$ is the branch of square root z chosen so that the branch cut is taken as the positive real axis, $[0, \infty)$, such that $\sqrt{-1} = i$



5. $\int_{\gamma} \frac{1+z}{e^z - e^{-z}} dz$ where γ is a positively (counter-clockwise) oriented parametrization of the circle $C_2 = \{z : |z - 2| = 1\}$

9. Prove that if f is an entire function and if there exist positive constants A and B such that $|f(z)| \leq A|z| + B$ for all $z \in \mathbb{C}$, then f is a linear function.

10. Let $D = \{z : |z| < 1\}$ and let $f \in \mathcal{A}(D)$ such that $f(D) \subset D$. Prove for $z \in D$ that $|f'(z)| \leq \frac{1}{1 - |z|}$. Hint. Suppose $z \in D$. Then, there exists a $\rho > 0$ such that $\overline{B(z, \rho)} \subset D$.

11. Let $D = \{z : |z| < 1\}$ and let $f \in \mathcal{A}(D)$ such that $f(D) \subset D$. Suppose that for each integer $n > 1$ that f satisfies $f(\frac{i}{n}) = -\frac{i}{n^3}$. Find the value of $f(\frac{1}{2} + \frac{1}{2}i)$.

12. State and prove Louiville's Theorem.

Classification Table for Problem 3

	open	closed	connected	polygonally path connected	compact	complete	bounded	region
A								
B								
C								
D								