Math 711: Lecture of November 7, 2007

Our current theory of test elements permits the extension of many results proved under other hypotheses, such as the condition that the ring under consideration be a homomorphic image of a Cohen-Macaulay ring, to the case of excellent local rings or, more generally, rings for which we have completely stable test elements (or completely stable big test elements, depending on whether the result being proved is for finitely generated modules or for arbitrary modules).

Here is one example:

Theorem (colon-capturing). *Let (R, m, K) be an excellent reduced equidimensional local ring of prime characteristic $p > 0$, and let $x_1, \ldots, x_{k+1}$ be part of a system of parameters. Let $I_k = (x_1, \ldots, x_k)R$. Then $I_k^* :_R x_{k+1} = I_k^*$. In particular, $I_k :_R x_{k+1} \subseteq I_k^*$.*

Proof. Note that R has a completely stable test element c. Suppose that $ux_{k+1} \in I_k$ but $u \in R - I_k$. This is also true when we pass to $\widehat{R}$, which is a homomorphic image of a regular ring and, hence, of a Cohen-Macaulay ring. Therefore, from the result on colon-capturing from p. 9 of the Lecture Notes from October 5, we have that $u \in (I_k\widehat{R})^*$, whence $cu^q \in (I\widehat{R})^{[q]} = I^{[q]}\widehat{R}$ for all q, and it follows that $cu^q \in I^{[q]}\widehat{R} \cap R = I^{[q]}$ for all q. Thus, $u \in I^*$.

Now suppose that $ux_{k+1} \in I_k^*$. Then $u^q x_{k+1}^q \in (I_k^*)^{[q]} \subseteq (I_k^{[q]})^*$ for all q, and so $cu^q x_{k+1}^q \in I_k^{[q]}$ for all q, and $cu^q \in I_k^{[q]} :_R x_{k+1}^q \subseteq (I_k^{[q]})^*$ by the result of the first paragraph applied to $x_1^q, \ldots, x_{k+1}^q$. Hence, $c^2u^q \in (I_k)^{[q]}$ for all q, so $u \in I_k^*$. The opposite conclusion is obvious. $\square$

A Noetherian ring is called *locally excellent* if its localization at every maximal ideal (equivalently, at every prime ideal) is excellent.

Corollary. *If R is weakly F-regular and locally excellent, then R is Cohen-Macaulay.*

Proof. Both weak F-regularity and the Cohen-Macaulay property are local on the maximal ideals of R. Hence, we may assume that R is local. Since weakly F-regular rings are normal, R is certainly equidimensional. Since colon-capturing holds for systems of parameters in R, the result is immediate. $\square$

We can also prove a global version of this Theorem above that is valid even in case the ring is not equidimensional. We need one additional fact.

Lemma. *Let (R, m, K) be an excellent local ring and let I be an ideal of R that has height at least k modulo every minimal prime of R. Then $I\widehat{R}$ has height at least k modulo every minimal prime of $\widehat{R}$.*

Proof. If $\mathfrak{p}_i$ is a minimal prime of R, $\mathfrak{p}_i\widehat{R}$ is a radical ideal and is the intersection of certain minimal primes $\mathfrak{q}_{ij}$. The intersection of all $\mathfrak{q}_{ij}$ is the same as the intersection of the $\mathfrak{p}_i\widehat{R}$. Since finite intersection commutes with flat base change, this is 0. Thus, it will suffice to show that the height of I is at least k modulo every $\mathfrak{q}_{ij}$. To this end, we can replace R by $R/\mathfrak{p}_i$. Thus, it is enough to show the result when R is an excellent local domain. In this case, $\widehat{R}$ is reduced and equidimensional. Any prime Q of $\widehat{R}$ containing $I\widehat{R}$ lies over a prime P of R containing I. The height of Q is at least the height of Q_0 where Q_0 is a minimal prime of $P\widehat{R}$. Hence, it suffices to show that if Q is a minimal prime of $P\widehat{R}$ then height (Q) = height (P). Since R and $\widehat{R}$ are equidimensional and catenary,

$$\text{height } P = \dim(R) - \dim(R/P) = \dim(\widehat{R}) - \dim(\widehat{R}/P\widehat{R}).$$

Since the completion of R/P is equidimensional,

$$\dim(\widehat{R}/P\widehat{R}) = \dim\left(\widehat{R}/Q\right).$$

Hence,

$$\text{height}(P) = \dim(\widehat{R}) - \dim\left(\widehat{R}/Q\right) = \text{height}(Q),$$

as required. $\square$

Theorem (colon-capturing). *Let R be a reduced Noetherian ring of prime characteristic $p > 0$ that is locally excellent and has a completely stable test element c . This holds, for example, if R is reduced and essentially of finite type over an excellent semilocal ring. Let $x_1, \ldots, x_{k+1}$ be elements of R. Let I_t denote the ideal $(x_1, \ldots, x_t)R$, $0 \leq t \leq k+1$. Suppose that the image of the ideal I_k has height k modulo every minimal prime of R, and that the image of the ideal $I_{k+1}R$ has height $k+1$ modulo every minimal prime of R. Then $I_k^* :_R x_{k+1} = I_k^*$.*

Proof. We first prove that $I_k :_R x_{k+1} \subseteq I_k^*$. The stronger conclusion then follows exactly as in the Theorem above because c is a test element.

If $x_{k+1}u \in I_k$ but $u \notin I_k^*$, we can choose q so that $cu^q \notin I_k^{[q]}$. This is preserved when we localize at a maximal ideal in the support of $(I_k^{[q]} + cu^q)/I_k^{[q]}$. We have therefore reduced to the case of an excellent local ring R_m. By the Lemma above, the hypotheses are preserved after completion, and we still have $cu^q \notin I_k^{[q]}S = (I_kS)^{[q]}$, where S is the completion of R_m. Since c is a test element in S, we have that $u \notin (I_kS)^*$. Since S is a homomorphic image of a Cohen-Macaulay ring, this contradicts the Theorem on colon-capturing from p. 9 of the Lecture Notes from October 5. $\square$

We next want to use the theory of test elements to prove results on persistence of tight closure.

Persistence

Let $\mathcal{R}$ (respectively, $\mathcal{R}_{\text{big}}$) denote the class of Noetherian rings S such that for every domain $R = S/P$, the normalization R' of R is module-finite over R and has the following two properties:

(1) The singular locus in R' is closed.

(2) For every element $c \in R' - \{0\}$ such that R'_c is regular, c has a power that is a test element (respectively, a big test element) in R'.

Of course, $\mathcal{R}_{\text{big}} \subseteq \mathcal{R}$, and $\mathcal{R}_{\text{big}}$ includes both the class of F-finite rings and the class of rings essentially of finite type over an excellent semilocal ring.

Theorem (persistence of tight closure). *Let R be in $\mathcal{R}$ (respectively,in $\mathcal{R}_{\text{big}}$). Let $R \to S$ be a homomorphism of Noetherian rings and suppose that $N \subseteq M$ are finitely generated (respectively, arbitrary) R-modules. Let $u \in N^*_M$. Then $1 \otimes u \in \langle S \otimes_R N\rangle^*_{S \otimes_R M}$.*

Proof. It suffices to prove the result after passing to $S/\mathfrak{q}_j$ as $\mathfrak{q}_j$ runs through the minimal primes of S. Therefore, we may assume that S is a domain. Let P denote the kernel of $R \to S$. Then P contains a minimal prime $\mathfrak{p}$ of R. Tight closure persists when we kill $\mathfrak{p}$ because the element c used in the tight closure test is not in $\mathfrak{p}$. Hence, we may make a base change to $R/\mathfrak{p}$, and so we may assume that $R \to S$ is a map of domains with kernel P. It suffices to prove that tight closure is preserved when we pass from R to R/P, since the injective map of domains $R/P \hookrightarrow S$ always preserves tight closure. Henceforth we may assume that S has the form R/P.

Choose a saturated chain of primes

$$(0) = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_h = P.$$

Then it suffices to show that tight closure persists as we make successive base changes to R/P_1, then to R/P_2, and so forth, until we reach $R/P_h = R/P$. Therefore, we need only prove the result when $S = R/P$ and P has height one.

Let R' be the normalization of R and let Q be a prime of R' lying over P. We have a commutative diagram

$$\begin{array}{ccc} R/P & \longrightarrow & R'/Q \\ \uparrow & & \uparrow \\ R & \longrightarrow & R' \end{array}$$

where the horizontal arrows are module-finite extensions and the vertical arrows are quotient maps. Tight closure is preserved by the base change from R to R' because it is an