

First Examination
Physics 5013, Mathematical Methods of
Physics

October 1, 2004

Instructions: Attempt all parts of this exam. If you get stuck on one part, assume an answer and proceed on. Do not hesitate to ask questions. Remember this is a closed book, closed notes, exam. *Good luck!*

1. Consider the function

$$f(z) = \frac{1}{z(1+z)}. \quad (1)$$

(a) If $|z| \ll 1$ show that f can be expanded in the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

and determine, by direct expansion, the coefficients a_n .

(b) What is the radius of convergence of the series found in part 1a?

(c) If $|z| \gg 1$ find another expansion of the form

$$f(z) = \sum_{m=-\infty}^{\infty} b_m z^m,$$

that is, determine all the coefficients b_m . What is the region in which this expansion converges?

(d) If K is a contour that encircles the origin once in the positive (counterclockwise) sense, show that by distorting the contour into a circle about the origin,

$$\oint_K z^n dz = 2\pi i \delta_{n,-1}, \quad n \text{ a positive or negative integer}$$

that is, the integral vanishes unless the power of z is -1 .

- (e) Consider a circle C of radius $1/2$ encircling the origin. Evaluate for f in (1), using Cauchy's theorem,

$$\oint_C f(z) dz,$$

if the circle is traversed once in the positive sense.

- (f) Now use the expansion found in part 1a to obtain the same result. Why is it legitimate to integrate term by term?
 (g) Now calculate for f in (1), using Cauchy's theorem,

$$\oint_{C'} f(z) dz,$$

where C' is a circle of radius 2 centered on the origin.

- (h) Verify this result by integrating term by term the series found in part 1c, using the result of part 1d.

2. Suppose the function $g(\omega)$ has N zeroes, at $\omega_0^{(i)}$, $i = 1, 2, \dots, N$, each of degree n_i , and M poles, at $\omega_\infty^{(j)}$, $j = 1, 2, \dots, M$, each of degree m_j in a certain region R in the complex ω plane. That is, the function can be written as

$$g(\omega) = \prod_{i=1}^N (\omega - \omega_0^{(i)})^{n_i} \prod_{j=1}^M (\omega - \omega_\infty^{(j)})^{-m_j} h(\omega),$$

where n_i and m_j are positive integers, and $h(\omega)$ is an analytic function having no zeroes in the region R . Then prove the following Principal Argument theorem: If C is a closed contour lying in R , which does not pass through any of the zeroes or poles,

$$\frac{1}{2\pi i} \oint_C \omega \frac{d}{d\omega} \log g(\omega) d\omega = \sum_i n_i \omega_0^{(i)} - \sum_j m_j \omega_\infty^{(j)},$$

where the sum extends over those zeros and poles which are contained inside of C . This formula is useful, for example, in evaluating zero-point energies.

3. The Riemann zeta function, for $\text{Re } s > 1$, is defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The gamma function, for $\text{Re } \alpha > 0$, may be taken to be defined by the Euler representation

$$\Gamma(\alpha) = \int_0^{\infty} \frac{dt}{t} t^{\alpha} e^{-t}.$$

(a) Show that $\Gamma(1) = 1$ and that $\Gamma(n+1) = n\Gamma(n)$, $n > 0$, so that for α equal to an integer, the Γ function may be identified with the factorial:

$$\Gamma(n+1) = n!.$$

(b) Now evaluate the integral

$$\int_0^{\infty} dy \frac{y^n}{e^y - 1}$$

by expanding the denominator in powers of $e^{-y} < 1$ and integrating term by term. Express the answer in terms of Γ and ζ functions.