Physics 5013. Homework 8
Due Tuesday, November 23, 2004

November 9, 2004

1. Let $\rho(x)$ be a positive function on the interval $a \leq x \leq b$. Consider the set of complex-valued functions $\{f\}$ defined on this same interval with the property that
$$\int_a^b dx\, \rho(x)\, |f(x)|^2 < \infty.$$
Prove that this set of functions is an inner product space, with vector addition defined by $(f+g)(x) = f(x) + g(x)$ and the inner product given by
$$\langle f, g\rangle = \int_a^b dx\, \rho(x)\, f^*(x) g(x).$$
Using the fact that $\mathcal{L}_2(a,b)$ is a Hilbert space, prove that this space is one also.

2. Using Green's theorem,
$$\int_V (d\mathbf{r}) \left[u\nabla^2 v - v\nabla^2 u\right] = \oint_S d\mathbf{S} \cdot \left[u\boldsymbol{\nabla} v - v\boldsymbol{\nabla} u\right],$$
where the volume V is bounded by the closed surface S, and $d\mathbf{S}$ is the outwardly directed surface element, find the three types of homogeneous boundary conditions which assure that the Laplacian operator ∇^2 is self-adjoint, in terms of the inner product
$$\langle u, v\rangle = \int_V (d\mathbf{r})\, u^*(\mathbf{r}) v(\mathbf{r}).$$

3. Find the eigenvalues and eigenfunctions of ∇^2 in two dimensions in the region $0 \le |\mathbf{r}| \le a$, subject to the boundary condition that all functions under consideration vanish at $|\mathbf{r}| = a$. (These might describe the normal modes of vibration of a circular drumhead.) [Hint: Solve the Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

by separating variables in polar coordinates.]

4. Suppose we have a second-order differential operator of the form

$$L = \frac{1}{f}\frac{d}{dx}\left(f\frac{d}{dx}\right) + q,$$

where f and q are functions of x. If y_1 and y_2 are independent solutions of

$$Ly = 0,$$

the Wronskian

$$\Delta(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

is different from zero. Prove that

$$\frac{d}{dx}\Delta = -\Delta \frac{d}{dx}\ln f,$$

and that

$$y_2(x) = \Delta(x_0) f(x_0) y_1(x) \int_{x_0}^{x} \frac{du}{f(u) y_1^2(u)},$$

where x_0 is a point at which

$$\begin{aligned} y_2(x_0) = 0, &\quad y_1(x_0) \neq 0, \\ f(x_0) \neq 0, &\quad y_1'(x_0) \neq 0. \end{aligned}$$

5. Recall that the Bessel functions of integer order are defined by

$$e^{(x/2)(z-1/z)} = \sum_{m=-\infty}^{\infty} z^m J_m(x),$$

or, with $x = kr$, $z = ie^{i\phi}$,

$$e^{ikr\cos\phi} = \sum_{m=-\infty}^{\infty} i^m e^{im\phi} J_m(kr).$$

Use this expression in the *two-dimensional* completeness statement for the functions

$$\frac{1}{2\pi}e^{i\mathbf{k}\cdot\mathbf{r}},$$

that is,

$$\int \frac{(d\mathbf{k})}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\mathbf{k}\cdot\mathbf{r}'} = \delta(\mathbf{r}-\mathbf{r}'),$$

where the right-hand side is a two-dimensional delta function, which in polar coordinates is

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{1}{r}\delta(r-r')\delta(\theta-\theta'),$$

and $(d\mathbf{k})$ is the two-dimensional integration element, which is correspondingly given in polar coordinates as

$$(d\mathbf{k}) = k\,dk\,d\alpha.$$

In this way derive the completeness property of the Bessel functions,

$$\int_0^\infty k\,dk\,J_m(kr)J_m(kr') = \frac{1}{r}\delta(r-r').$$