Physics 5013. Homework 5
Due Friday, October 28, 2011

October 16, 2011

1. Show that the function $1/z^2$ represents the analytic continuation of the function defined by the series

$$\sum_{n=0}^{\infty}(n+1)(z+1)^n, \quad |z+1|<1,$$

into the domain consisting of all points in the z plane except $z=0$.

2. Show that

$$e^{uz+v/z} = a_0 + a_1 z + a_2 z^2 + \ldots + \frac{b_1}{z} + \frac{b_2}{z^2} + \ldots,$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta}\cos[(u-v)\sin\theta - n\theta]\, d\theta, \\ b_n &= \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta}\cos[(v-u)\sin\theta - n\theta]\, d\theta. \end{aligned}$$

3. Show that the function

$$f(z) = (z^2-1)^{1/2}$$

is single valued when it is defined with the branch line running along the real axis from -1 to $+1$. [Hint: Consider the net phase change in f when the branch line is encircled once.]

4. By integrating $\frac{e^{\pm aiz}}{e^{2\pi z}-1}$ around a rectangle whose corners are 0, R, $R+i$, i (the rectangle being indented at 0 and i) and letting $R \to \infty$, show that
$$\int_0^\infty \frac{\sin ax}{e^{2\pi x}-1}dx = \frac{1}{4}\frac{e^a+1}{e^a-1} - \frac{1}{2a},$$
a result due to Legendre.

5. Show that if $a > 0$, $b > 0$,
$$\int_0^\infty e^{a\cos bx}\sin(a \sin bx)\frac{dx}{x} = \frac{\pi}{2}(e^a - 1).$$

6. Show that
$$\int_0^{\pi/2} \frac{a \sin 2x}{1 - 2a\cos 2x + a^2} x\, dx = \begin{cases} \frac{\pi}{4}\log(1+a), & -1 < a < 1, \\ \frac{\pi}{4}\log(1+a^{-1}), & a^2 > 1. \end{cases}$$

7. Evaluate
$$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}(1+x^2)^2}$$
by using a contour which encircles the branch line given in Problem 3, and closed by a circle at infinity. Equivalently, consider a contour of two parts: one that just encloses the branch line from $z = -1$ to $z = +1$ and another being a circle about the origin of very large radius. Between these two contours, the function
$$f(z) = \sqrt{1-z^2}$$
is analytic.

8. Recall the generating function defining the Bernoulli numbers:
$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$
Show that
$$B_n = \frac{n!}{2\pi i}\oint_{C_0} \frac{z}{e^z-1}\frac{dz}{z^{n+1}},$$
where C_0 is a circle about the origin with radius $|z| < 2\pi$. From this integral find B_0, B_1 directly. By distorting C_0 into C, an infinite circle

about the origin (and hence crossing an infinite number of poles!), show that for n even, $n \geq 2$,

$$B_n = -\frac{(-1)^{n/2} 2\, n!}{(2\pi)^n} \zeta(n),$$

where

$$\zeta(n) = \sum_{p=1}^{\infty} p^{-n}.$$

Use the residue theorem to evaluate the following integrals:

9.

$$\int_0^{2\pi} \frac{d\theta}{1 + a \sin^2 \theta}, \quad a > 0.$$

10.

$$\int_0^{\infty} \frac{dx \, \sin x}{x(x^2 + a^2)}, \quad a > 0.$$

11.

$$\int_0^{\infty} \frac{x^{2a-1}}{1 + x^2}, \quad 0 < a < 1.$$

12.

$$P \int_0^{\infty} dx \, \frac{\sqrt{x}}{x^2 - a^2}, \quad a > 0.$$