Physics 5013
Mathematical Methods of Physics
First Examination
October 10, 2001

Instructions: Work all problems and show all your steps. If you have any questions, ask the instructor. The points carried by each part are indicated; do not spend undue time on any one problem. Remember, this is a closed-book examination, so you may not consult books or notes. **Good Luck!**

Note: Unless otherwise specified, all variables are complex.

1. This problem explores certain properties of the exponential function, and of its inverse, the logarithm.

[5] (a) The exponential function is defined by the power series

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

Use the root test to determine the radius of convergence of this series. For what complex values of x does e^x possess singularities?

[10] (b) Using the power-series definition of the exponential function, establish the conditions under which

$$e^{x+y} = e^x e^y.$$

[5] (c) Let π be the smallest positive number for which

$$e^{2i\pi} = 1.$$

Show then that the exponential function is periodic with period $2\pi i$,

$$e^{x+2\pi i n} = e^x, \quad n \text{ an integer}.$$

[5] (d) Evaluate the derivative of e^x,

$$\frac{d}{dx}e^x.$$

[10] (e) If $x = e^y$, define the inverse function, the logarithm, by $y = \ln x$. Is the logarithm single-valued? If not, give all the values of $\ln x$ for a given x. What is $\ln 1$?

[5] (f) Evaluate the derivative of the logarithm,

$$\frac{d}{dx}\ln x.$$

[10] (g) Compute all the derivatives of $\ln(1 + x)$, in particular, give a closed-form expression for

$$\frac{d^n}{dx^n}\ln(1+x)\bigg|_{x=0}.$$

[10] (h) Thus determine the power series expansion of $\ln(1 + x)$ for small x. What is the radius of convergence of this series? Why?

[20] 2. Consider the integral

$$\oint_C dz\, z^\alpha$$

where C is a circular contour which is centered on the origin, encircling it once in a counterclockwise sense, and α is a complex number. Evaluate the integral explicitly for all values of α. From the result, what can you say about the values of α for which z^α is analytic? In particular, what can you say about the case when α is a negative integer?

3. In this problem you may use Cauchy's theorem to evaluate the integrals.

[10] (a) Consider the integral

$$\oint_{C_0} \frac{dz}{e^z - 1},$$

where C_0 is a circle about the origin with radius less than 2π, encircling the origin once in a positive sense. What is the value of this integral?

[10] (b) Now consider the integral

$$\oint_{C_n} \frac{dz}{e^z - 1},$$

where C_n is a positively-oriented circle about the origin with radius ρ in the interval $2\pi n < \rho < 2\pi(n+1)$. Does this integral have a different value? Why? If it is different, evaluate it.